Polynomials: Understanding Practice 3

Goal: Build a better understanding of division of polynomials and using remainder theorem and factor theorem with polynomials in $\mathbb{Z}[X]$.

When we work with pairs of numbers, we often want to find the GCD particularly when we add them together or we are looking to reduce the quotient. There is an algorithm to fins a GCD for integers that translates to polynomials naturally called the **Euclidean Algorithm**.

Example: Find $\gcd(1073, 666)$

$$1073 = 1 \cdot 666 + 407$$
$$666 = 1 \cdot 407 + 259$$
$$407 = 1 \cdot 259 + 148$$
$$259 = 1 \cdot 148 + 111$$
$$148 = 1 \cdot 111 + 37$$
$$111 = 3 \cdot 37 + 0$$
$$\Rightarrow \gcd(1073, 666) = 37$$

Note $1073 = 29 \cdot 37$ and $666 = 2 \cdot 3^2 \cdot 37$

We take the larger number and divide it by the smaller number and add the remainder:

$$\frac{1073}{666} = 1 + \frac{407}{666}$$
$$\Rightarrow 1073 = 1 \cdot 666 + 407$$

Then we repeat with the divisor and the remainder (which is 666 and 407) until we get a remainder of 0.

The last non-zero remainder is the gcd!

1. Find $\gcd(4182, 1599)$

$4182 = 2 \cdot 1599 + 984$

$1599 = 1 \cdot 984 + 615$

$984 = 1 \cdot 615 + 369$

$615 = 1 \cdot 369 + 246$

$369 = 1 \cdot 246 + 123$

$246 = 2 \cdot 123 + 0$

$\Rightarrow \gcd(4182, 1599) = 123$

☆ $4182 = 34 \cdot 123$

$1599 = 13 \cdot 123$

2. Explain the analog of the algorithm where we have two polynomials $a(x)$ and $b(x)$ and degree of a is greater than or equal to the degree of b.

take $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)} \Rightarrow a(x) = q(x) \cdot b(x) + r(x)$

Repeat with $\frac{b(x)}{r(x)}$ until the remainder is 0

3. Find $\gcd(x^2+7x+6,\ x^2-5x-6)$

$$\begin{array}{r} 1 \\ x^2-5x-6 \overline{)\, x^2+7x+6} \\ -(x^2-5x-6) \\ \hline 12x+12 \end{array} = r(x)$$

$$\begin{array}{r} \frac{1}{12}x - \frac{1}{2} \\ 12x+12 \overline{)\, x^2-5x-6} \\ -(x^2+x) \\ \hline -6x-6 \\ -6x-6 \\ \hline 0 \end{array}$$

$x^2+7x+6 = 1(x^2-5x-6) + 12x+12$

$x^2-5x-6 = (\frac{1}{12}x-\frac{1}{2})(12x+12) + 0$

$\Rightarrow$ gcd is $(12x+12)$ ☆ But $12x+12$ does not divide either over $\mathbb{Z}[x]$ so the gcd is just $\underline{\underline{x+1}}$ (factor the 12 out and discard)

4. Find $\gcd(a(x),\ b(x))$ where:

$$\cancel{a(x) = x^5 - 5x^4 - 6x^3 + 76x^2 - 152x + 96}$$
$$\cancel{b(x) = 2x^3 - 15x^2 + 31x - 12}$$

$$a(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$$
$$b(x) = x^3 - 7x^2 + 15x - 9$$

$$\begin{array}{r} x + 1 \\ x^3 - 7x^2 + 15x - 9 \enclose{longdiv}{x^4 - 6x^3 + 9x^2 + 4x - 12} \\ -(x^4 - 7x^3 + 15x^2 - 9x) \\ \hline x^3 - 6x^2 + 13x - 12 \\ -(x^3 - 7x^2 + 15x - 9) \\ \hline x^2 - 2x - 3 \end{array}$$

$$\begin{array}{r} x - 5 \\ x^2 - 2x - 3 \enclose{longdiv}{x^3 - 7x^2 + 15x - 9} \\ -(x^3 - 2x^2 - 3x) \\ \hline -5x^2 + 18x - 9 \\ -(-5x^2 + 10x + 15) \\ \hline 8x - 24 \end{array}$$

$$a(x) = (x+1)\,b(x) + x^2 - 2x - 3$$
$$b(x) = (x-5)(x^2 - 2x - 3) + 8x - 24$$
$$x^2 - 2x - 3 = 8(x-3)(x+1)\cdot\tfrac{1}{8} + 0$$

$\Rightarrow$ last nonzero remainder is $8x - 24 = 8(x-3)$

$$\Rightarrow \gcd = x - 3$$

5. Find $\gcd(A(x), B(x))$ where:

$$\cancel{A(x) = 2x^5 - x^4 + 8x^3 - 4x^2 + 6x - 3}$$
$$\cancel{B(x) = 3x^4 - 4x^3 + 10x^2 - 12x + 3}$$

$$A(x) = x^5 - 8x^3 - 6x^2 + 7x + 6$$

$$B(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$$

$$\begin{array}{r|l} & x + 2 \\ x^4 - 2x^3 - 3x^2 + 8x - 4 & x^5 - 8x^3 - 6x^2 + 7x + 6 \\ & -(x^5 - 2x^4 - 3x^3 + 8x^2 - 4x) \\ & 2x^4 - 5x^3 - 14x^2 + 11x + 6 \\ & -(2x^4 - 4x^3 - 6x^2 + 16x - 8) \\ & -x^3 - 8x^2 - 5x + 14 = R_1(x) \end{array}$$

$$\begin{array}{r|l} & -x + 10 \\ -x^3 - 8x^2 - 5x + 14 & x^4 - 2x^3 - 3x^2 + 8x - 4 \\ & -(x^4 + 8x^3 + 5x^2 - 14x) \\ & -10x^3 - 8x^2 + 22x - 4 \\ & -(-10x^3 - 80x^2 - 50x + 140) \\ & 72x^2 + 72x - 144 = R_2(x) \end{array}$$

$$\begin{array}{r|l} & \frac{1}{72}(-x - 7) \\ 72(x^2 + x - 2) & -x^3 - 8x^2 - 5x + 14 \\ & -(-x^3 - x^2 + 2x) \\ & -7x^2 - 7x + 14 \\ & -(-7x^2 - 7x + 14) \\ & 0 \end{array}$$

$$\Rightarrow A(x) = (x+2)B(x) + R_1(x)$$
$$B(x) = (10 - x)R_1(x) + R_2(x)$$
$$R_1(x) = -\frac{1}{72}(x+7)R_2(x) + 0$$

$\Rightarrow \gcd = R_2(x)$ but in $\mathbb{Z}[x]$ it is $\boxed{x^2 + x - 2}$

Remainder theorem says if $p(x)$ is divided by $x - a$ then its remainder is just $p(a)$. That is $p(a) = r$.

1. If the polynomial $p(x)$ is divided by $x - 3$ and the remainder is -2, what point must be on the curve p?

$(3, -2)$

$p(3) = -2$

$p: 3 \mapsto -2$

2. If the polynomial $p(x)$ is divisible by $x + 4$, what point must be on the curve p?

$p(-4) = 0$ $\quad p: -4 \mapsto 0$

$(-4, 0)$

3. If we know $p(x)$ is divisible by $x + 4$ then what does its factored form look like?

$p(x) = (x+4) \cdot q(x)$

4. If a polynomial is divisible by $x + 1$, but it has a remainder of 3 when divided by $x - 1$ and a remainder of -1 when divided by $x + 3$, what points must the curve pass through and how can we write it in factored form?

$p(-1) = 0$ $\quad p(1) = 3$ $\quad p(-3) = -1$

pass thru $(-1, 0)$; $(1, 3)$; $(-3, -1)$

$p(x) = (x+1)\, q(x)$

$p(1) = 3 = 2q(1)$ $\quad p(-3) = -2\,q(-3) = -1$

$\Rightarrow q(1) = \frac{3}{2}$ $\quad \Rightarrow q(-3) = \frac{1}{2}$

5. For the previous polynomial, if it is a parabola what would the equation to the parabola be? Express the polynomial as $A \cdot q(x)$ where $q(x) \in \mathbb{Z}[X]$ and $A \in \mathbb{Q}$.

let $q^*(x) = (x-a)A \Rightarrow q^*(1) = \frac{3}{2} \quad q^*(-3) = \frac{1}{2}$

$\Rightarrow \frac{3}{2} = (1-a)A \qquad \frac{1}{2} = (-3-a)A$

$A = \frac{3}{2(1-a)} = \frac{-1}{2(3+a)}$

$\Rightarrow 3(3+a) = (a-1)$

$9 + 3a = a - 1 \Rightarrow a = -5 \Rightarrow A = \frac{1}{4}$

$p(x) = \frac{1}{4}(x+1)(x+5)$

$= \frac{1}{4}x^2 + \frac{3}{2}x + \frac{5}{4}$

6. What if the polynomial was a cubic instead? Find one such cubic and explain why there are infinitely many cubic polynomials that satisfy these conditions.

$p(x) = (x+1)(ax^2 + bx + c) \qquad p(-1) = 0 \quad p(1) = 3 \quad p(-3) = -1$

① $3 = 2(a+b+c)$

② $-1 = -2(9a - 3b + c)$

$\Rightarrow$ Two equations and three unknowns should give us ∞ solutions

set b or $c = 0$ (we'll use $b=0$)

$\Rightarrow 3 = 2a + 2c$

$-1 = -18a - 2c$

$+$

$2 = -16a \Rightarrow a = \frac{-1}{8}$ and $c = \left(3 + \frac{1}{4}\right)\frac{1}{2} = \frac{13}{8}$

$p(x) = (x+1)\frac{1}{8}(-x^2 + 13) = -\frac{1}{8}(x^3 + x^2 - 13x - 13)$