Factoring Polynomials

KNOW	**DO**	**UNDERSTAND**
How to find the remainder using remainder theorem. How to test a factor using factor theorem	Factor polynomials Use long division to divide polynomials	*Function Characteristics*: That the output of a polynomial at any point $x = c$ is the remainder after being divided by $x - c$. Polynomials in $\mathbb{Z}[X]$ can be prime and have similar properties to integers (gcd/lcm, etc).

Vocab & Notation

- Set of polynomials with integer coefficients: $\mathbb{Z}[X]$
- Irreducible polynomial
- Factor Theorem
- Integer Root Theorem & Rational Root Theorem
- Remainder Theorem

We are motivated to factor polynomials just like we factor integers, but to do so we need to be able to divide polynomials. As we are motivated to factor like integers, we are only going to look at polynomials whose coefficients are integers.

Definition: Let $\mathbb{Z}[X]$ be the set of polynomials with integer coefficients. From now on, any polynomial I think about will be in this set and we will factor in this set.

$3x^2 - 7 \in \mathbb{Z}[X]$ $\frac{1}{2}x^3 - 3 \notin \mathbb{Z}[x]$

For example, factor 4199.

try 2? ends in 9 $\Rightarrow\Leftarrow$

try 3? $4+1+9+9 = 23$ not divisible by 3

try 5? ends in 9 $\Rightarrow\Leftarrow$

try 7? 4200 is divisible by 7
4199 has no chance

try 11? $4-1+9-9 = 3$ not divisible by 11

$$\begin{array}{r} 323 \\ 13\,\overline{)\,4199} \\ -\,3900 \\ \hline 299 \\ -\,260 \\ \hline 39 \\ 39 \\ \hline 0 \end{array}$$

$\Rightarrow$ no remainder

We want to take these ideas and help us factor:

$$p(x) = x^4 - x^3 - 7x^2 + 13x - 6$$

$$\Rightarrow p(x) = (x-a)\,q(x) = (x-a)(x^3 + \dots + c)$$
$$= x^4 + \dots - ac$$

Idea 1:

$p(a) = \overset{0}{\cancel{(a-a)}}\,q(a)$
$= 0$

Idea 2

a must divide the constant term

Theorem: The **factor theorem** states that if $(x-a)$ divides $p(x)$ then $p(a)=0$.

But how do we find a?

Theorem: The **integer root theorem** states that if $(x-a)$ is a factor of $p(x)=x^n+\cdots+C$ then a is a factor of C.

*Note: there is a larger version of this theorem called the **rational root theorem**: given the polynomial $p(x)=Ax^n+\cdots+C$, if $x=\frac{a}{b}$ is a zero we will have that a is a factor of C and b is a factor of A.

So, let's factor this polynomial!

① Integer Root Thm

$$p(x)=x^4-x^3-7x^2+13x-6 \rightarrow \text{possible factors}$$

$\pm1, \pm2, \pm3, \pm6$

$\boxed{(x-1)}$, $(x+1)$, $(x-2)$ ... $(x+6)$

possible factors of $p(x)$

② use factor Thm

$p(1) = 1-1-7+13-6 = 0$

$\Rightarrow$ $x-1$ is a factor!

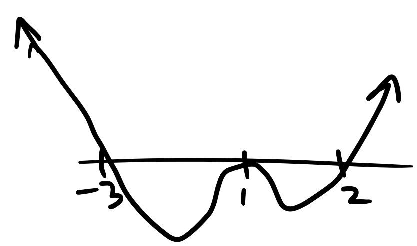
$$\frac{p(x)}{x-1} = x^3-7x+6$$

$$\begin{array}{r l}
 & x^3 \qquad -7x \quad +6 \\
x-1 \,\big) & x^4-x^3-7x^2+13x-6 \\
 & -(x^4-x^3) \\
\hline
 & -7x^2+13x-6 \\
 & -(-7x^2+7x) \\
\hline
 & 6x-6 \\
 & -(6x-6) \\
\hline
 & 0
\end{array}$$

$\Rightarrow$ No remainder!

$$\frac{4199}{13}$$

$$\begin{array}{r l}
 & 3 \\
13 \,\big) & 4\cdot10^3+1\cdot10^2+9\cdot10+9 \\
 & -(3\cdot10^3+9\cdot10^2) \\
\hline
 & 2\cdot10^2+9\cdot10+9
\end{array}$$



$$p(x)=(x-1)\underbrace{(x^3-7x+6)}_{q(x)} = (x-1)^2(x-2)(x+3)$$

$q(1) = 1-7+6 = 0 \Rightarrow (x-1)$ is a factor again!

Practice: Factor the polynomials and sketch them. State the intervals the polynomial is positive.

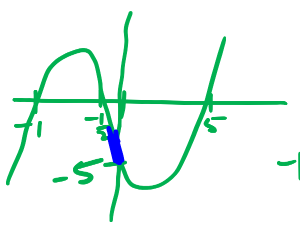
$2x^3 - 9x^2 - 6x + 5$

$\pm 1, \pm 5$

$p(-1) = -2 - 9 + 6 + 5 = 0$

$$
\begin{array}{r}
2x^2 - 11x + 5 \\
x+1 \,\big)\, 2x^3 - 9x^2 - 6x + 5 \\
-(2x^3 + 2x^2) \\
\hline
-11x^2 - 6x + 5 \\
-(-11x^2 - 11x) \\
\hline
5x + 5 \\
-5x + 5 \\
\hline
0
\end{array}
$$

$\Rightarrow p(x) = (x+1)(2x+1)(x-5)$



$p(x) > 0$ when $-1 < x < -\frac{1}{2}$ or $x > 5$

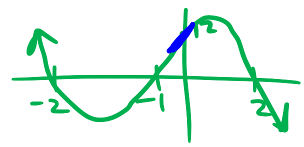
$-x^3 - x^2 + 8x + 12$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$p(-2) = 8 - 4 - 16 + 12 = 0$

$$
\begin{array}{r}
-x^2 + x + 2 \\
x+2 \,\big)\, -x^3 - x^2 + 8x + 12 \\
-(-x^3 - 2x^2) \\
\hline
x^2 + 8x + 12 \\
-(x^2 + 2x) \\
\hline
6x + 12
\end{array}
$$

$\Rightarrow p(x) = (x+2)(-1)(x-2)(x+1)$



$p(x) > 0$ when $x < -2$ or $-1 < x < 2$

$x^3 + 2x^2 - 3x - 6$

$\pm 1, \pm 2, \pm 3, \pm 6$

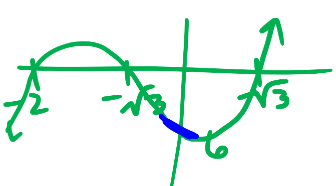
$p(-2) = -8 + 8 + 6 - 6 = 0$

$$
\begin{array}{r}
x^2 - 3 \\
x+2 \,\big)\, x^3 + 2x^2 - 3x - 6 \\
-(x^3 + 2x^2) \\
\hline
-3x - 6 \\
-(-3x - 6) \\
\hline
0
\end{array}
$$

$\Rightarrow p(x) = (x+2)(x^2+3)$ ☆ $x^2+3 = (x+\sqrt{3})(x-\sqrt{3})$

But these are not in $\mathbb{Z}[x]$

$\Rightarrow x^2+3$ is prime



$p(x) > 0$ when $x \in (-2, \sqrt{3}) \cup (\sqrt{3}, \infty)$

$x^4 - 3x^2 - 4$

use long division OR

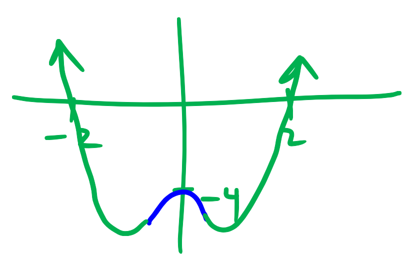
note $x^4 - 3x^2 - 4 = y^2 - 3y - 4$

$= (y-4)(y+1)$

$= (x^2-4)(x^2+1)$

$= (x-2)(x+2)(x^2+1)$

☆ x^2+1 is prime and has no zeros



$p(x) > 0$ when $|x| > 2$

We can also divide polynomials by polynomials that are not factors and end up with a remainder.

Example: $435 \div 7$

$$
\begin{array}{r}
62 \\
7\overline{)435} \\
-420 \\
\hline
15 \\
14 \\
\hline
1
\end{array}
\quad \Rightarrow \quad \frac{435}{7} = 62 + \frac{1}{7}
$$

Example: Simplify the following quotient:

$$\frac{x^4 - 2x^2 + 5x + 1}{x + 2} = \frac{p(x)}{q(x)} = Q(x) + \frac{r(x)}{q(x)}$$

$$
\begin{array}{r}
x^3 - 2x^2 + 2x + 1 \\
x+2\overline{)x^4 - 2x^2 + 5x + 1} \\
-(x^4 + 2x^3) \\
\hline
-2x^3 - 2x^2 + 5x + 1 \\
-(-2x^3 - 4x^2) \\
\hline
2x^2 + 5x + 1 \\
-(2x^2 + 4x) \\
\hline
x + 1 \\
x + 2
\end{array}
$$

$$\frac{p(x)}{q(x)} = x^3 - 2x^2 + 2x + 1 + \frac{-1}{x+2}$$

remainder $= -1$

$$p(-2) = 16 - 8 - 10 + 1 = -1$$

Theorem: The **remainder theorem** says if $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$

$$\frac{p(x)}{q(x)} = Q(x) + \frac{r(x)}{q(x)} \Rightarrow p(x) = Q(x) \cdot q(x) + r(x) \Rightarrow p(a) = \cancel{Q(a)} \cdot 0 + r$$

(a, r) is on p

Practice: Determine the remainder of the following quotient using long division and verify using remainder theorem.

$$\frac{-x^3 + 2x^2 + 5x - 2}{x - 3} = \frac{p(x)}{x - 3}$$

$$
\begin{array}{r}
-x^2 - x + 2 \\
x-3\overline{)-x^3 + 2x^2 + 5x - 2} \\
-(-x^3 + 3x^2) \\
\hline
-x^2 + 5x - 2 \\
-(-x^2 + 3x) \\
\hline
2x - 2 \\
-(2x - 6) \\
\hline
4
\end{array}
$$

$$p(3) = -27 + 18 + 15 - 2 = 4$$

$$\Rightarrow \boxed{-x^2 - x + 2 + \frac{4}{x-3}}$$

Practice Problems: 3.2 page 124 – 125 # 1-3, 5-11, 14, 15, 17, C1, C2
3.3 page 133 – 135 # 1-8, 15, 16, C1