Area as Net Change

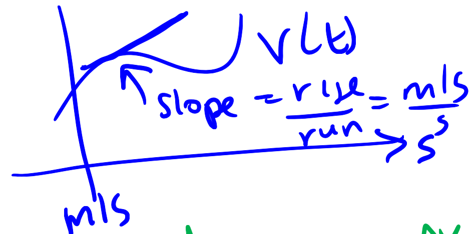
Goal:
- Understands that the area under a curve of some rate of change will be a measure of the total change over an interval of time.
- Can use integration with applications of $f(t)$ where f has units of something per unit time (velocity, acceleration, flow, etc)

Terminology:
- Net Change

Given the velocity of a particle (m/s) along the x-axis is

$$v(t) = \frac{t^2}{20} - \frac{1}{t^2+1}$$

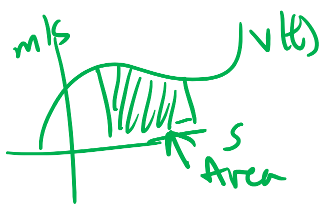
Create an equation for the position function and the acceleration function. Explain how the process of building the equation gives you the proper units.



$$a(t) = \frac{d}{dt}v(t) = \frac{t}{10} + \frac{2t}{(t^2+1)^2} \quad m/s^2$$

$$\frac{d}{dt}v = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$x(t) = \int v(t)\,dt = \frac{t^3}{60} - \arctan t + C \quad m$$



$v \times t$

$$\int_0^x v(t)\,dt = \lim_{\Delta t \to 0} \sum v \Delta t$$

What is the total displacement of the particle if $t \in [0,6]$? What is the total distance travelled by the particle on that interval? Note that $x(0) = 1$

Total displacement $\quad x(6) - x(0) = \int_0^6 v(t)\,dt = \frac{6^3}{60} - \arctan 6$

$$\Rightarrow x(t) = \int_0^t v(x)\,dx + 1$$

dummy

Total distance $\quad \int_0^6 |v(t)|\,dt = \int_0^2 -v(t)\,dt + \int_2^6 v(t)\,dt$

At what time on the interval $0 < t < 6$ does the particle change direction?

find when $x'(t) = 0 = v(t) = \frac{t^2}{20} - \frac{1}{t^2+1}$

$t = 2$

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In general, when we are using a function $f(t)$ whose input units are “time” and output units are “things” we can say what the units of the derivative and integral are

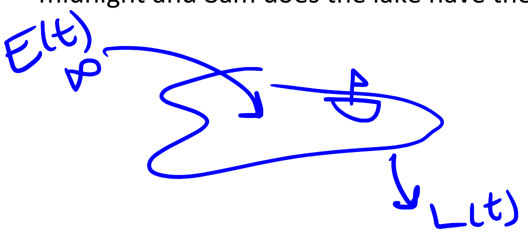
1. $\frac{d}{dt}f(t)$ has units things / unit time m/s km/h^2

   $\frac{\Delta f}{\Delta t}$ people / day \$ / sec

2. $\int f(t)\,dt$ has units Rate: things * unit time $m/s * s$ people/day * day

   $\sum f \Delta t$ thing / time * time \$/sec * sec

Example: Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight. How many fish enter the lake between midnight and 5am? At what time between midnight and 8am does the lake have the greatest number of fish?



Total fish = fish/hr × hr

$= \int E(t)\,dt$

$\text{Total fish}_{in} = \int_0^5 E(t)\,dt$

$= \int_0^5 20 + 15\sin\left(\frac{\pi t}{6}\right) dt$

$= 20t - 15\cos\left(\frac{\pi t}{6}\right)\cdot\frac{6}{\pi}\Big|_0^5 = 153$ fish

$\frac{d}{dt}\left(F(t) = \int_0^t \left(E(x) - L(x)\right) dx + F_0\right)$ + −

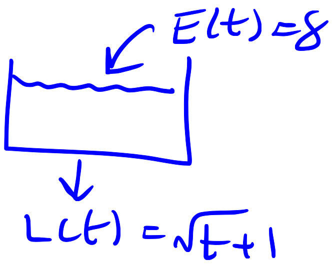
$F'(t) = E(t) - L(t) = 0$

$20 + 15\sin\left(\frac{\pi t}{6}\right) - 4 - 2^{0.1t^2} = 0$

graph → $t = 6.204$ ⇒ 6:12 am

Practice: Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \leq t \leq T$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

How much water is in the tank after 30 minutes?



$$W(t) = \int_0^t [E(\tau) - L(\tau)]\,d\tau + 30$$

$$W(30) = 8\tau - \frac{2}{3}(\tau+1)^{3/2}\Big|_0^{30} + 30$$

$$= 155.6 \text{ gallons}$$

At what time does the tank have the most water in it?

$$W'(t) = E(t) - L(t) = 0$$

$$\Rightarrow 8 - \sqrt{t+1} = 0$$

$$t = 63 \quad \text{max}$$

$+ \quad -$

63

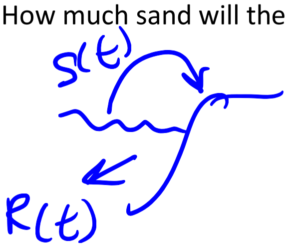
If the tank empties after T minutes, what is the value of T?

$$W(T) = 0 = \int_0^T (E(\tau) - L(\tau))\,d\tau + 30$$

graph find $T = 148.5$ hr

Practice: The tide removes sand from a beach at a rate modeled by the function $R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right)$. A pumping station adds sand to the beach at a rate modeled by the function $S(t) = \frac{15t}{1+3t}$. Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

How much sand will the tide remove from the beach during this 6-hour period?



$$\int_0^6 R(t)\,dt = 2t - \frac{125}{4\pi}\cos\left(\frac{4\pi t}{25}\right)\Big|_0^6$$

$$= 31.8 \text{ yards}^3$$

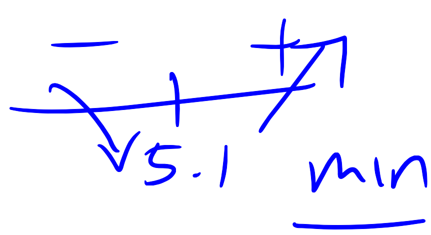
How much sand is on the beach after t hours?

$$\text{Sand}(t) = \int_0^t \left[\frac{15\tau}{1+3\tau} - 2 - 5\sin\left(\frac{4\pi\tau}{25}\right)\right] d\tau + 2500$$

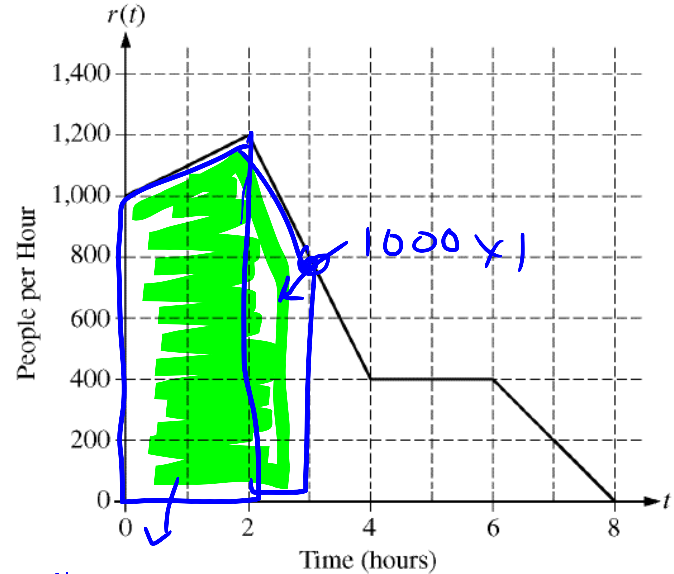
At what time will the amount of sand on the beach be minimal?

$$\text{Sand}'(t) = \frac{15t}{1+3t} - 2 - 5\sin\left(\frac{9\pi t}{25}\right) = 0$$

$\rightarrow$ graph $\quad t = 5.1$ hr



Practice:
There are 700 people in line for a popular amusement-park ride when the ride begins operation in the morning. Once it begins operation, the ride accepts passengers until the park closes 8 hours later. While there is a line, people move onto the ride at a rate of 800 people per hour. The graph to the right shows the rate, $r(t)$, at which people arrive at the ride throughout the day. Time t is measured in hours from the time the ride begins operation.



1100 x 2

How many people arrive at the ride between $t = 0$ and $t = 3$?

$$\int_0^3 r(t)\,dt = 3200 \text{ people}$$

At what time in the line for the ride the longest? How many people are in line at that time?

$$L(t) = \int_0^t (r(\tau) - 800)\,d\tau + 700$$

$$L'(t) = r(t) - 800 = 0$$

@ $t = 3$ $\quad$ $+ \; 3 \; -$ $\quad$ max

Write, but do not solve, an equation involving an integral expression of r whose solution gives the earliest time t at which there is no longer a line for the ride.

$$L(t) = 0 \Rightarrow \int_0^t (r(\tau) - 800)\,d\tau + 700 = 0$$

first such t

($t = 5.5$ if solved)

Practice: The temperature outside a house during a 24-hour period is given by $C(t) = 27 - 5\cos\left(\frac{\pi t}{12}\right)$, $0 \leq t \leq 24$, where $C(t)$ is measured in degrees Celsius and t is measured in hours.

Find the average temperature when $t \in [6, 14]$

$$\text{avg (Temp)} = \frac{1}{8}\int_6^{14} C(t)\,dt$$

$$= \frac{1}{8}\left[27t - 5\sin\left(\frac{\pi t}{12}\right)\cdot\frac{12}{\pi}\right]_6^{14} = 30.6^\circ C$$

An air conditioner cooled the house whenever the outside temperature was at or above 26 degrees Celsius. For what values of t was the air conditioner cooling the house?

$$C(t) \geqslant 26$$

$$\Rightarrow \quad 1 - 5\cos\left(\frac{\pi t}{12}\right) \geqslant 0$$

$$t = 5.23,\ 18.77$$

The cost of cooling the house accumulates at the rate of $0.20 per hour for each degree the outside temperature exceeds 26°C. What was the total cost, to the nearest cent, to cool the house for this 24–hour period?

$$\text{Total cost} = \int_{t_1}^{t_2} 0.2\left(1 - 5\cos\left(\frac{\pi t}{12}\right)\right)dt$$

$$= 0.2\left[t - \frac{60}{\pi}\sin\left(\frac{\pi t}{12}\right)\right]_{5.23}^{18.77}$$

$$= \$10.19$$

Practice Problems: 7.1 # 21-24, 28-30	
Textbook Readings: 7.1 page 363-370	
Workbook Practice: page 350-358	
Next Class: Volume	