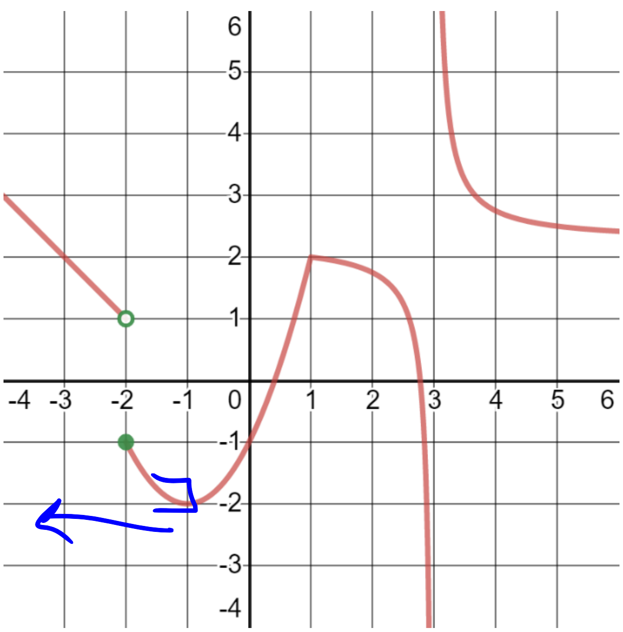
Critical Points and Mean Value Theorem

Goal: • Can determine critical points and intervals of increasing/decreasing. • Can derive Rolle's Theorem and MVT • Can use MVT to show that a function with positive derivative is increasing.
Terminology: • Critical Point • Global and Local Extremum • Increasing/Decreasing • Extreme Value Theorem • Rolle's Theorem • Mean Value Theorem

We are starting our final derivative unit on applications of derivatives and we are going to begin by doing some curve analysis.

We begin with some function f, not necessarily differentiable or continuous over $\mathbb{R}$. We want to know where does f achieve a max or minimum value (important if we want to optimize the function) and where is f increasing and decreasing (important to know how a small change in x will affect $f(x)$).



When does f achieve a max or minimum? Without a graph how would you identify it?

When is f increasing or decreasing? Without a graph how would you identify it?

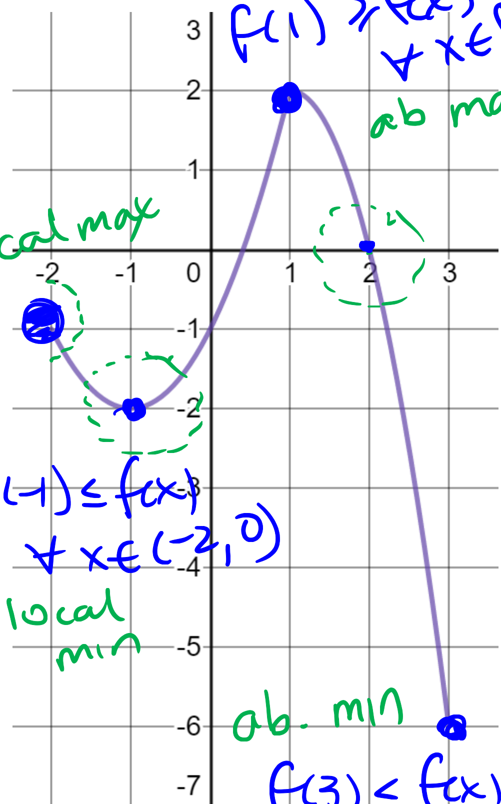
we are decreasing when $x \le -1$

increasing on $[-1, 1]$

decreasing on $[1, 3) \cup (3, \infty)$

Definition: A *critical point* is when $f'(c)=0$ or $f'(c)$ is undefined.

Something interesting could happen



$f(1) \geq f(x)$ $\forall x \in [-2,3]$

ab max

local max

$f(-1) \leq f(x)$ $\forall x \in (-2,0)$

local min

ab. min

$f(3) \leq f(x)$ $\forall x \in [-2,3]$

Definition: Given a function $f: D \to \mathbb{R}$ then $c \in D$ is an *absolute maximum* if

$f(c) \geq f(x)$ $\forall x \in D$

for all

Definition: Given a function $f: D \to \mathbb{R}$ then $c \in D$ is a *local maximum* if

$f(c) \geq f(x)$ $\forall$ x in some open interval around c. in D

Find critical points and value of f on endpoints

Practice: Find all extrema for the following function on the indicated domain

$f(x) = \frac{1}{x} + \ln x, \qquad 0.5 \leq x \leq 4$

$x^{-1} + \ln x$

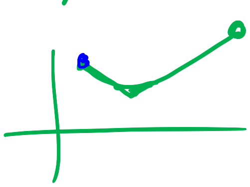
$-\frac{1}{x^2} + \frac{1}{x} = \frac{1-x}{x^2}$

CP. $x=1$, $\cancel{x=0}$

$f(0.5) = 1.306\ldots$ local max

$f(1) = 1$ abs. min

$f(4) = 1.636\ldots$ abs. max



$g(x) = x^{\frac{2}{3}}, \qquad -3 \leq x < 1$

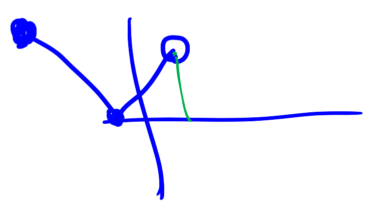
$g'(x) = \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3\sqrt[3]{x}}$

CP: $x=0$

$g(-3) = 2.08$ abs max

$g(0) = 0$ abs min

$g(1) = 1$ nothing



Theorem: The extreme value theorem says the function f will achieve a maximum and minimum value on it's domain if

The domain is closed $[a,b]$

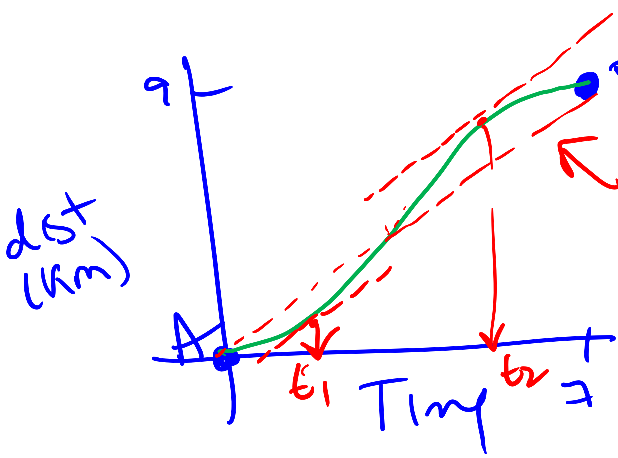
and f is continuous on $[a,b]$

We are going to use the extreme value theorem to prove on of the most important theorems in calculus, but first a classic anecdote:

"Police have two radar controls at a highway, say at kilometre 11 and at kilometre 20. The speed limit is 70 km/h. They measure a truck going through the first control, at 11:11am, at 65 km/h, and going through the second control at 11:17am, at 67 km/h. They issue a speeding ticket. Why?"

The total distance was 9 km and total time was 7 min $\Rightarrow$

$$\text{avg speed} = \frac{9\text{ km}}{7\text{ min}} \times \frac{60\text{ min}}{1\text{ hr}} = 77\text{ km/h}$$



@ t_1 and t_2 the slope = avg slope

* path was continuous on $[0,7]$ and differentiable

Theorem: The Mean Value Theorem says that if $f: [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) then …

$\exists\ c \in (a,b)$ such that

there exists

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x}$$

Theorem: Rolle's thoerem say that if $f: [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) AND $f(a) = f(b)$ then …

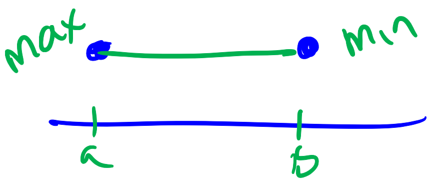
☆ $\exists\ c \in (a,b)$ such that

$$f'(c) = 0$$

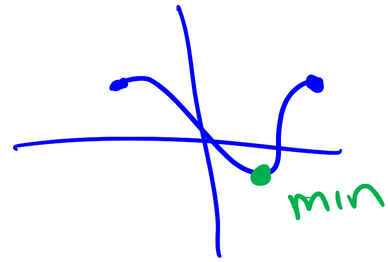
Case I:

if $f(a)$ is the max/min

then we have a line whose slope is 0 everywhere



Case II:



$f'(x)$ exist $\forall\ x \in (a,b)$ and we have a min (or max)

Then $f'(c) = 0$ at the max (or min) $x = c$

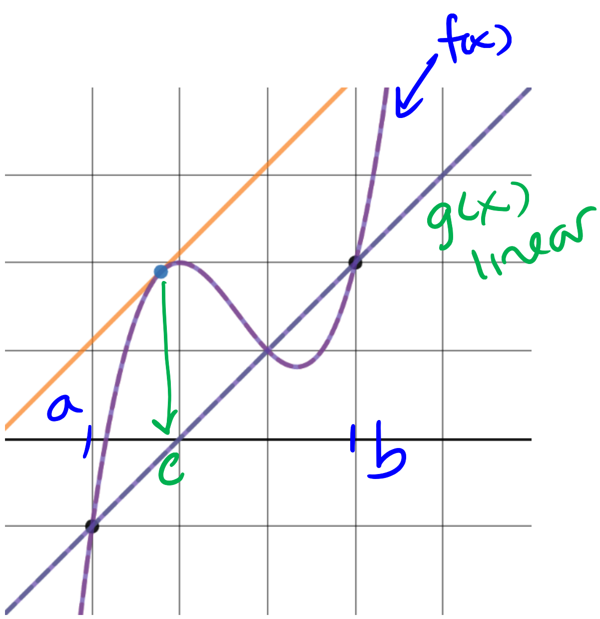
Proof of MVT:

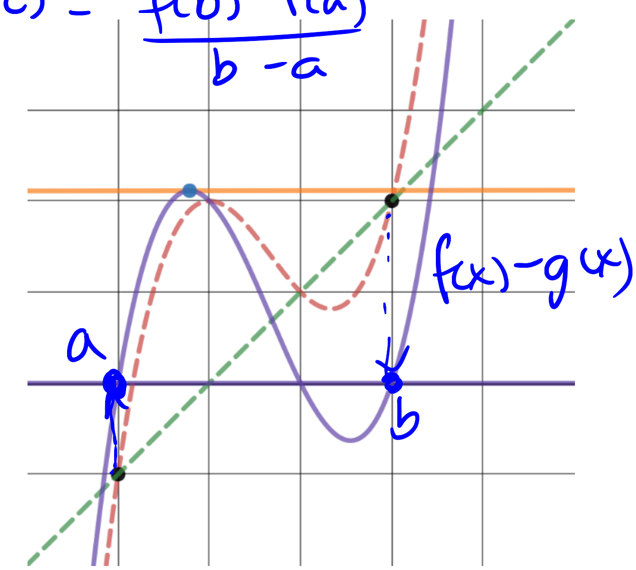
☆ make $g(x)$ such that $g(a) = f(a)$, $g(b) = f(b)$

$F(x) = f(x) - g(x)$ ☆ $F(a) = 0 = F(b)$ we can use Rolle's

so $\exists\ c \in (a,b)$ such that $F'(c) = 0 = f'(c) - g'(c)$

$$f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}$$





I mentioned that MVT is one of the most important theorems and it is at the heart of a lot of impotant results, we'll look at two here but understanding MVT and feeling comfortable to apply it will make you a better calculus student.

Definition: A function is (strictly) increasing on the interval I if $\forall\, x, y \in I$ with $x < y$, then we get $f(x) < f(y)$.

Corollory: A function is (strictly) increasing on the closed interval $[a, b]$ if f is differentiable on (a, b), continuous on $[a, b]$ and $f'(c) > 0\ \forall\, c \in (a, b)$

Proof: From MVT if we pick $x, y \in [a,b]$ then on $[x,y]$ we can find $c \in [x,y]$ such that

$$f'(c) = \frac{f(y)-f(x)}{\underbrace{y-x}_{>0}} > 0 \Rightarrow f(y)-f(x) > 0$$

$$f(y) > f(x)$$

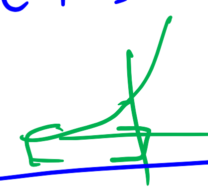
We can also use MVT to prove inequalities. For example, a simpler version of the inequality on the last assignment

$$x + 1 \le e^x \text{ when } x \le 0$$

Proof:

$f(x) = e^x$ on $x \le 0$ make $[a, 0]$ $\exists\, c \in (a, 0)$ s.t

$$f'(c) = \frac{f(0)-f(a)}{0-a} = 0 < e^c = \frac{1-e^a}{-a} \le 1$$



$$1 - e^a \le -a \Rightarrow \boxed{1 + a \le e^a}$$

Practice: Prove that $\cos x \ge 1 - x$ when $x \ge 0$

Practice Problems: 4.1: # 1-6 and 11-30 (select), 49, 50, 52 4.2: # 1-14 (select), 15-18, 21-24, 35-42, 47, 56, 57	
Textbook Readings: 4.1 page 177-183 and 4.2 page 186-188	
Workbook Practice: page 177-181, 234-237	
Next Day: First and Second Derivative Tests	