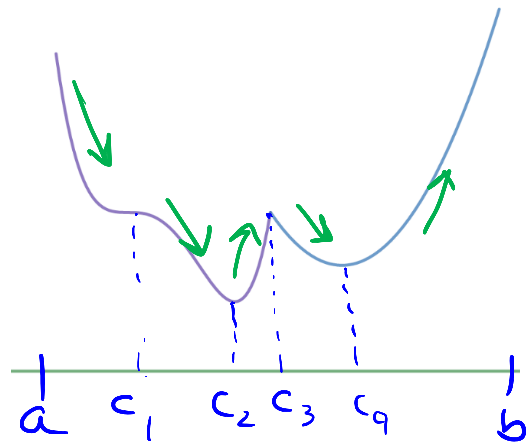
First and Second Derivative Test

Goal: • Can use the first derivative test and second derivative test to find extrema • Can determine concavity by considering f' or f''
Terminology: • First Derivative Test • Concavity • Inflection Point • Second Derivative Test

We are going to consider that $f: [a, b] \to \mathbb{R}$ is continuous and we want to find a way to determine if a critical point is a maximum or a minimum. Note that f need not be differentiable on (a, b).

x	a	c_1	c_2	c_3	c_4	b
$f'(x)$	$-$	0 $-$	0 $+$	$\emptyset$ $-$	0	$+$
Behaviour of f	↘	↘	min ↗	max ↘	min	↗



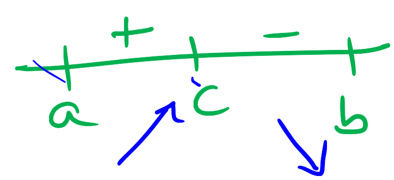
Recall: If f is differentiable on (a, b), continuous on $[a, b]$ and $f'(x) > 0 \ \forall\, x \in (a, b)$ then f is (strictly) increasing on $[a, b]$.

Proof: If $x < y$ Start $\Rightarrow f(x) < f(y)$ $[x, y]$ where $a \le x < y \le b$

Then $\exists\, c \in (x, y)$ s.t. $f'(c) = \dfrac{f(y) - f(x)}{y - x} > 0$, $y - x > 0$

$\Rightarrow f(y) - f(x) > 0$

Theorem: The first derivative test says that if f is continuous on $[a, b]$ and we have that $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b), then $x = c$ is a local maximum.



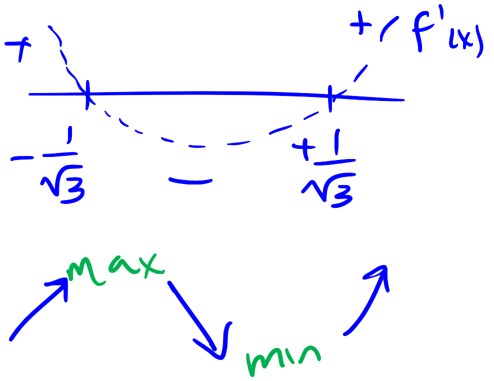
Proof:
since $f'(x) > 0$ on (a, c)
we are increasing on $[a, c]$
$f(c) \geq f(x) \quad \forall\, x \in [a, c]$
likewise $f(c) \geq f(x) \quad \forall\, x \in [c, b] \Rightarrow f(c)$ is a local max.

Practice: Use the first derivative test to find all extrema of the following functions:

$f(x) = x^3 - x, \quad x \in [-1, 2]$

$g(x) = 2x^4 + 4x^3 - 1, \quad x \in [-2, 1]$

$f'(x) = 3x^2 - 1$;



$f(2) = 6$

$f(1/\sqrt{3}) =$

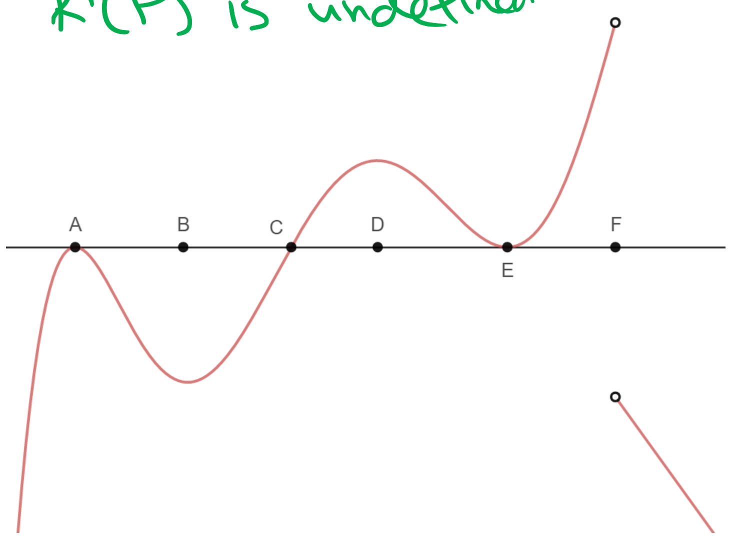
$f(-1/\sqrt{3}) =$

$f(-1) = 0$

$h(x) = e^{\cos x}, \quad x \in [-2, 4]$

The following is a graph of $\frac{dk}{dx}$ for some continuous k

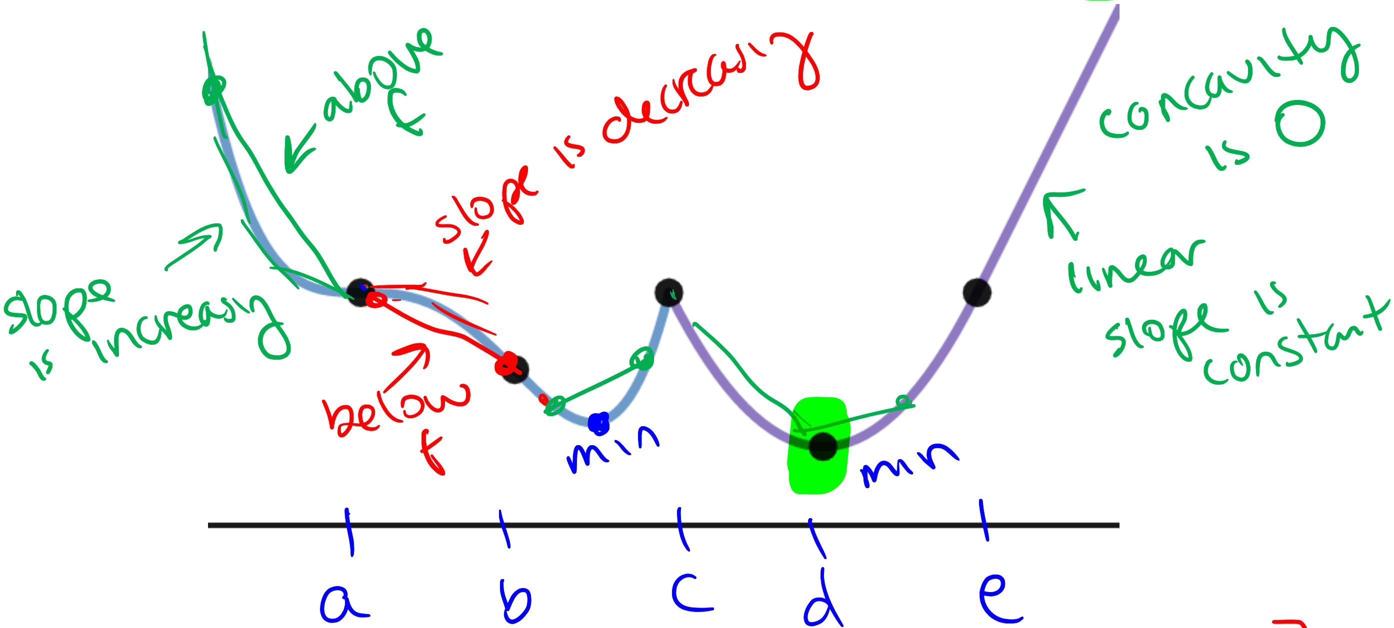
$k'(F)$ is undefined



Definition: A differentiable function is *concave up* on (a, b) if f' is (strictly) increasing on (a, b). Note the open intervals as f' may not exist at the endpoints.

**If f is differentiable on $[a, b]$ then concave up implies f' is (strictly) increasing on $[a, b]$

Note that if the function is twice differentiable, we can say that if f' is increasing on $[a, b]$, then $f''(x) > 0\ \forall\, x \in (a, b)$

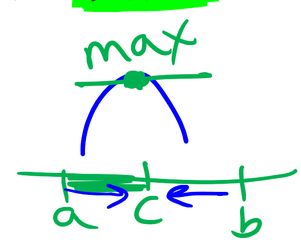


concave up $x \le a$, $[b, c) \cup (c, e]$

concave down on $[a, b]$

Theorem: The second derivative test says that if f is twice differentiable in an open interval (a, b), and $f'(c) = 0$, and $f''(x) < 0\ \forall\, x \in (a, b)$, then $f(c)$ is a local maximum on $[a, b]$.

Proof: use MVT on $\underset{g(x)}{f'(x)}$ on $[a, c]$



we can find some $d \in (a, c)$ st

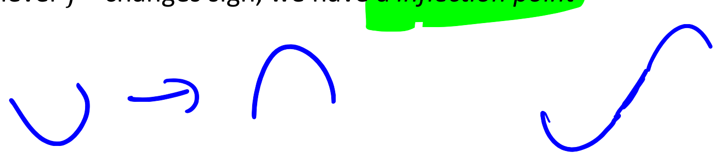
$$g'(d) = f''(d) = \frac{\overset{0}{f'(c)} - f'(a)}{\underset{>0}{c - a}} < 0 \Rightarrow -f'(a) < 0$$

$f'(a) > 0$

Do the same on $[c, b]$ $\Rightarrow f'(b) < 0$

by 1st derivative test c is a local max

Definition: Whenever f'' changes sign, we have *a inflection point*



Practice: Determine the intervals the functions are concave up/down

$$\ell(x) = \frac{x}{x^2+1}$$

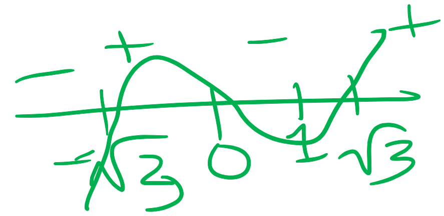
$$\ell'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2} = L(x) \Rightarrow L'(x) > 0$$

$$\ell''(x) > 0$$

$$\ell''(x) = \left(-2x(1+x^2)^2 - 2\cancel{(1+x^2)}(2x)(1-x^2)\right)\frac{1}{(x^2+1)^4} > 0$$

$$= -2x\left((1+x^2)+2(1-x^2)\right)C$$

$$= -2x(3-x^2)C$$



concave down $x \le -\sqrt{3}$ and $x \in [0, \sqrt{3}]$; up $x \in [-\sqrt{3}, 0]$ and $x \geq \sqrt{3}$

$$m(x) = x^3 e^x$$

$m''(x) = ?$

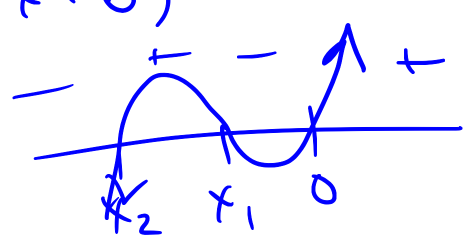
$$m'(x) = 3x^2e^x + x^3e^x = (3x^2+x^3)e^x$$

$$m''(x) = (6x+3x^2)e^x + (3x^2+x^3)e^x$$

$$= e^x(x^3+6x^2+6x)$$

$$= xe^x(x^2+6x+6)$$

$$x = \frac{-6 \pm \sqrt{36-24}}{2} = -3 \pm \sqrt{3}$$



concave up $x \geq 0$ and $x \in [x_2, x_1]$

and down $x \leq x_2$ and $x \in [x_1, 0]$

Practice: Use the second derivative test to find the local max and minimums. Assume that n is continuous.

x	$x < A$	A	$A < x < B$	B	$B < x < C$	C	$x > C$
$n''(x)$	Positive	0	Negative	undefined	Positive	0	Positive
$n'(x)$	0 For some $a < A$	5	0 For some $a_b \in (A, B)$	undefined	0 For some $b_c \in (B, C)$	0	Positive

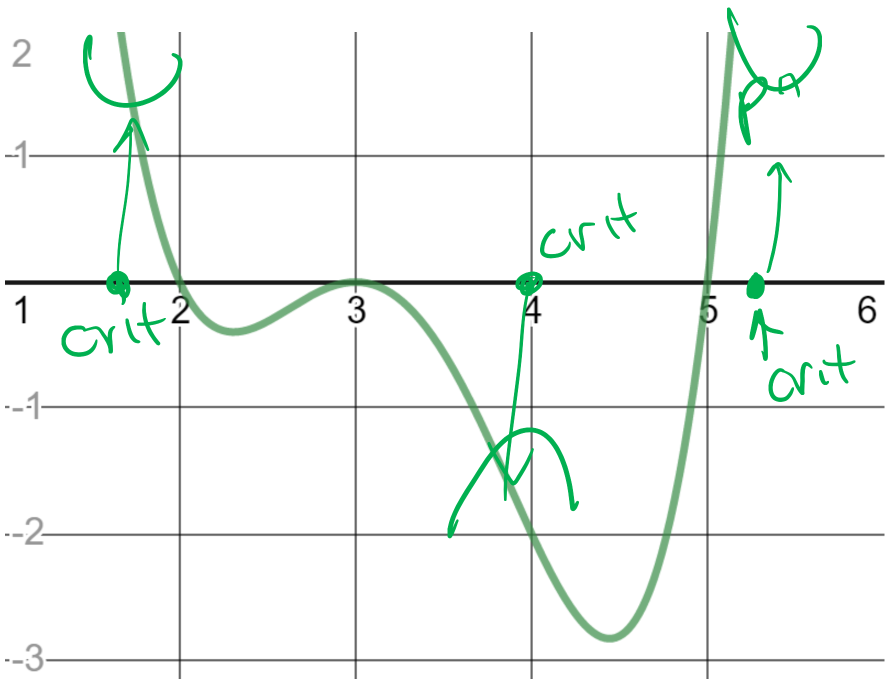
slope

@ $x=a$ local min

local max at $x=a_b$

local min @ $x=b_c$

Consider the graph of p'' below. If $p'(x) = 0$ when $x = 1.7, 4,$ and 5.3 then determine where the extrema of p occur and the type of extrema.



min @ $x = 1.7$ and 5.3

max @ $x = 4$

Practice Problems: 4.3: # 5-12 (select), 13-28 (select and use technology), 41-44, 50, 52	
Textbook Readings: 4.3 page 194-200	
Workbook Practice: page 182-193	
Next Day: Curve Sketching	