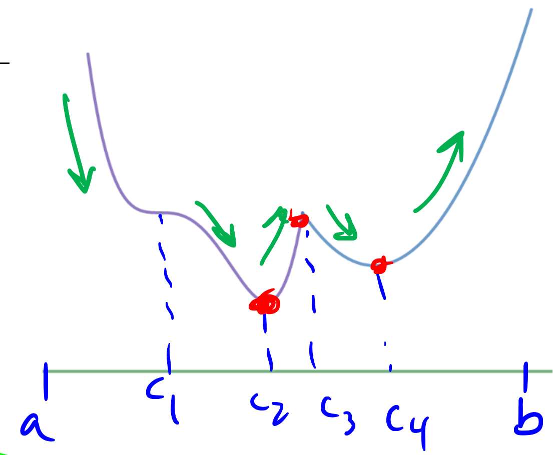
First and Second Derivative Test

Goal:
- Can use the first derivative test and second derivative test to find extrema
- Can determine concavity by considering f' or f''

Terminology:
- First Derivative Test
- Concavity
- Inflection Point
- Second Derivative Test

We are going to consider that $f: [a, b] \to \mathbb{R}$ is continuous and we want to find a way to determine if a critical point is a maximum or a minimum. Note that f need not be differentiable on (a, b).

x	a		c_1		c_2		c_3		c_4		b
$f'(x)$		$-$	0	$-$	0	$+$	$\not\exists$	$-$	0	$+$	
Behaviour of f		↘		↘	min	↗	max	↘	min	↗	



Recall: If f is differentiable on (a, b), continuous on $[a, b]$ and $f'(x) > 0\ \forall\, x \in (a, b)$ then f is (strictly) increasing on $[a, b]$.

$(x < y) \Rightarrow f(x) < f(y)$

Proof:

on $[x,y]$ $\exists\, c$ s.t. $f'(c) = \dfrac{f(y) - f(x)}{y - x} > 0$, $\quad y - x > 0$

$f(y) - f(x) > 0 \Rightarrow f(y) > f(x)$

Theorem: The first derivative test says that if f is continuous on $[a, b]$ and we have that $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b), then $x = c$ is a local maximum.

increas. decreas.

Proof:

on $[a,c]$ f is increasing

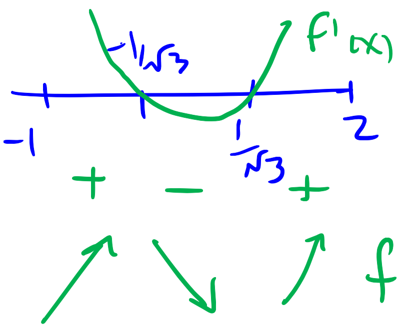
$\Rightarrow f(c) \geq f(x) \quad \forall\, x \in [a,c]$

on $[c,b)$ f is decreasing so $f(c) \geq f(x) \quad \forall\, x \in [c,b]$

Practice: Use the first derivative test to find all extrema of the following functions:

$f(x) = x^3 - x, \quad x \in [-1, 2]$

$f'(x) = 3x^2 - 1 = 0 \quad @ \ x = \pm \frac{1}{\sqrt{3}}$



max @ $f(-\frac{1}{\sqrt{3}}) = 0.4$

min @ $f(\frac{1}{\sqrt{3}}) = -0.4$ (abs)

max @ $f(2) = 6$ (abs)

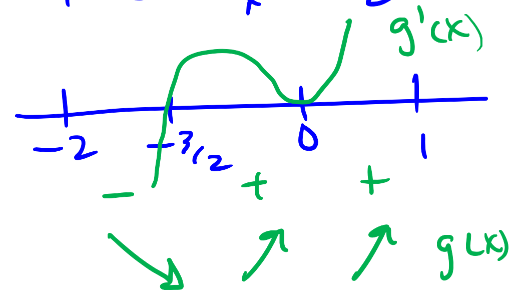
min @ $f(-1) = 0$

$g(x) = 2x^4 + 4x^3 - 1, \quad x \in [-2, 1]$

$g'(x) = 8x^3 + 12x^2$

$0 = 4x^2(2x + 3)$

@ $x = 0 \quad x = -3/2$



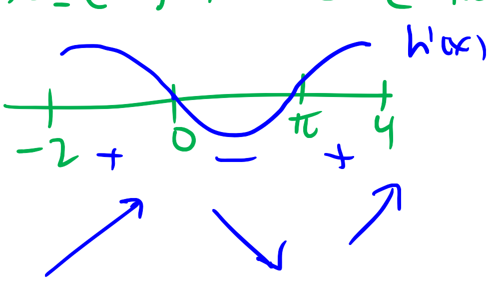
max @ $f(-2) = -1$

min @ $f(-3/2)$ (abs)

max @ $f(1) = 5$ (abs)

$h(x) = e^{\cos x}, \quad x \in [-2, 4]$

$h'(x) = -e^{\cos x} \sin x = 0 \quad @ \ x = 0, \pi$



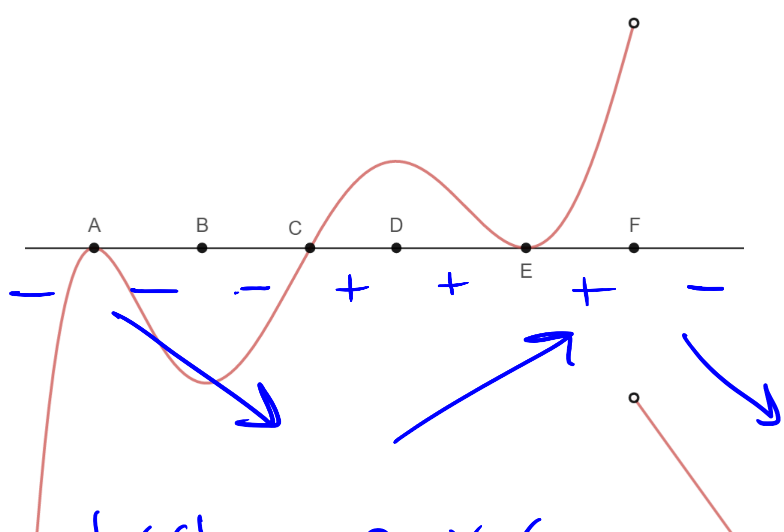
min @ $f(-2) = 0.6$

max @ $f(0) = e$ (abs)

min @ $f(\pi) = \frac{1}{e} = 0.4$ (abs)

max @ $f(4) = 0.5$

The following is a graph of $\frac{dk}{dx}$ for some continuous k



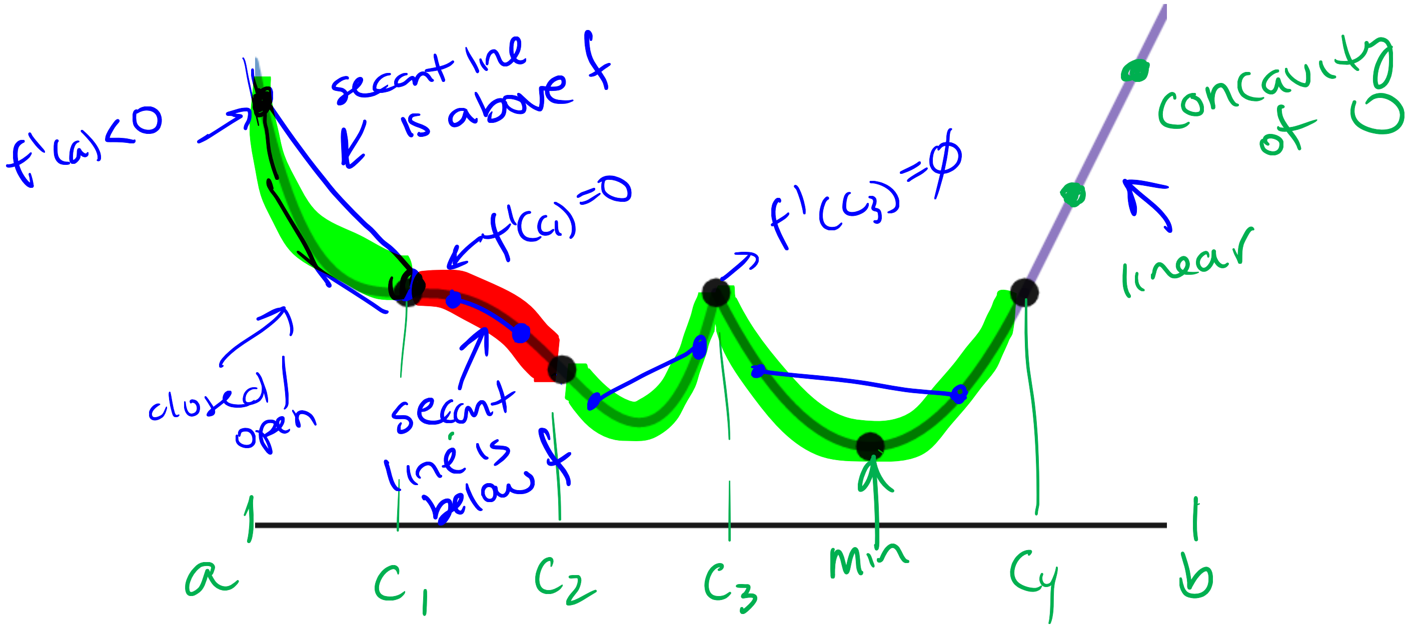
local min @ $x = C$

local max @ $x = F$

Definition: A differentiable function is *concave up* on (a, b) if f' is (strictly) increasing on (a, b). Note the open intervals as f' may not exist at the endpoints.

**If f is differentiable on $[a, b]$ then concave up implies f' is (strictly) increading on $[a, b]$

Note that if the function is twice differentiable, we can say that if f' is increasing on $[a, b]$, then $f''(x) > 0 \;\forall\, x \in (a, b)$

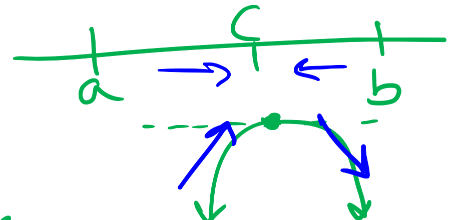


concave up $a \le x \le c_1$; $c_2 \le x < c_3$; $c_3 < x \le c_4$

concave down $c_1 \le x \le c_2$

Theorem: The second derivative test says that if f is twice differentiable in an open interval (a, b), and $f'(c) = 0$, and $f''(x) < 0 \;\forall\, x \in (a, b)$, then $f(c)$ is a local maximum on $[a, b]$.

Proof:



$f' = g$ on $[a, c]$ has MVT properties

so find some $d \in (a, c)$ s.t

$$g'(d) = \frac{g(c) - g(a)}{c - a} \equiv f''(d) = \frac{\cancel{f'(c)} - f'(a)}{c - a} < 0 \Rightarrow -f'(a) < 0$$

$f'(c) = 0$, $c - a > 0$

$f'(a) > 0$

★ show that on $[c, b]$ $f'(b) < 0$

$\Rightarrow f(c)$ is a local max by 1st derivative test.

Definition: Whenever f'' changes sign, we have a *inflection point*

Practice: Determine the intervals the functions are concave up/down

$$\ell(x) = \frac{x}{x^2+1}$$

$\ell''(x) > 0$

$$m(x) = x^3 e^x$$

$m''(x) > 0$

Practice: Use the second derivative test to find the local max and minimums. Assume that n is continuous.

concavity

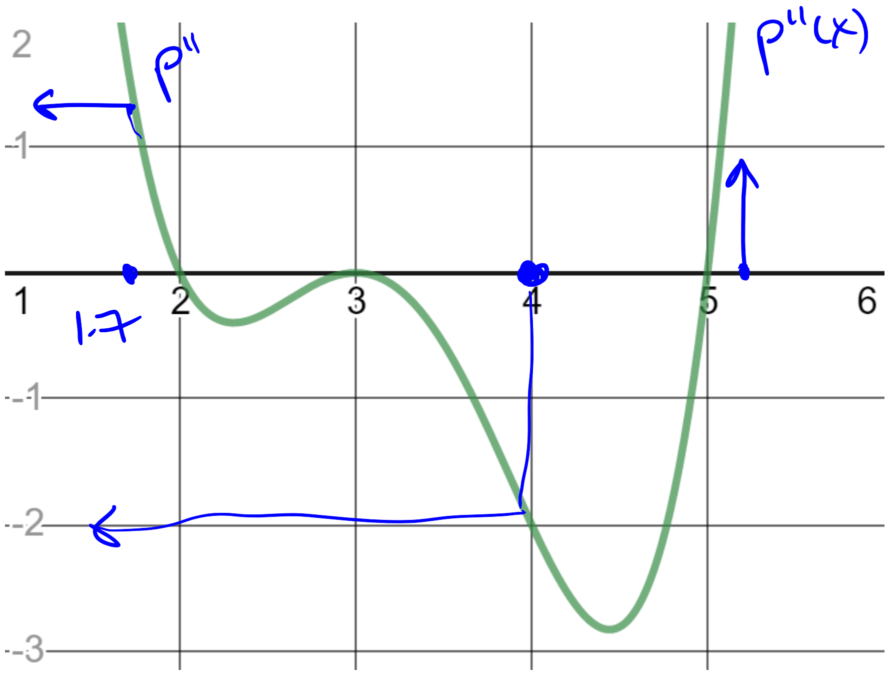
slope

x	$x < A$	A	$A < x < B$	B	$B < x < C$	C	$x > C$
$n''(x)$	Positive	0	Negative	undefined	Positive	0	Positive
$n'(x)$	0 For some $a < A$	5	0 For some $a_b \in (A, B)$	undefined	0 For some $b_c \in (B, C)$	0	Positive

@ $x = a$ we have a local min and $x = b_c$

$x = a_b$ we have a local max

Consider the graph of p'' below. If $p'(x) = 0$ when $x = 1.7, 4,$ and 5.3 then determine where the extrema of p occur and the type of extrema.



$p'(1.7) = 0$

$p''(1.7) > 0$

$\Rightarrow x = 1.7$ is a local min

$p'(4) = 0$

$p''(4) < 0$

$\Rightarrow x = 4$ is local max

@ $x = 5.3$ is a local min

Practice Problems: 4.3: # 5-12 (select), 13-28 (select and use technology), 41-44, 50, 52	
Textbook Readings: 4.3 page 194-200	
Workbook Practice: page 182-193	
Next Day: Curve Sketching	