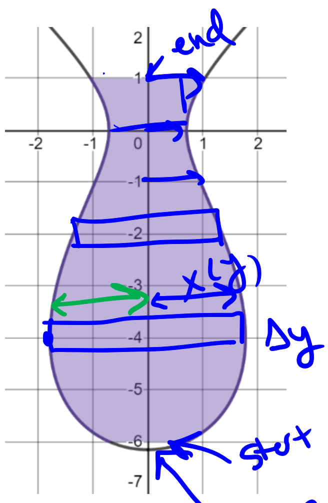
Volumes Part 1: Cross Sections and Rotations

Goal:
• Can determine the volume of solids using geometric slices • Can derive the volume equations to pyramids and spheroids • Can determine the volume of solids after rotation around the x or y axis
Terminology: • None

Determine the area of the shaded region given the curve is

$$\frac{1}{4}\left((2x)^2 = \frac{y^3}{3} + 2y^2 + 2\right)$$



$$A = \lim_{n\to\infty} \sum_{n=1}^{n} 2x(y)\cdot \Delta y$$

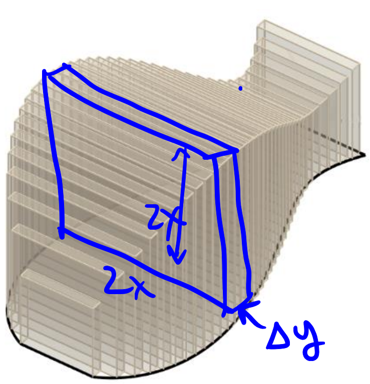
a function of y

$$= \int_{-6}^{1} 2x(y)\, dy$$

$$= \int_{-6}^{1} \sqrt{\frac{y^3}{3} + 2y^2 + 2}\; dy = 20 \text{ guess}$$

$$= 18.1$$

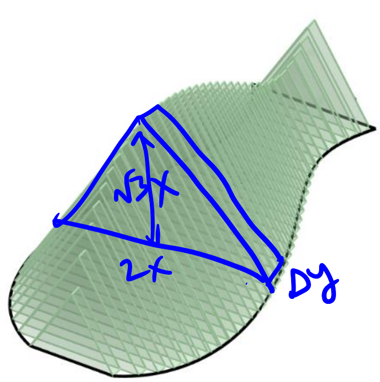
Consider a few differnet ways of making a vase out of this region



squares

$$V = \lim_{n\to\infty} \sum (2x)^2\, \Delta y$$

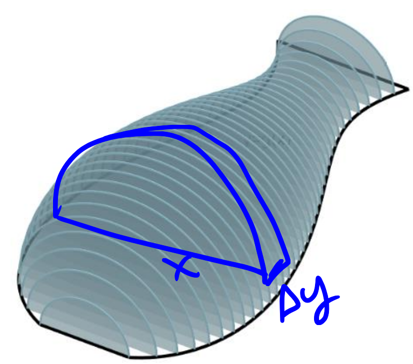
$$= \int_{-6}^{1} \left(\frac{y^3}{3} + 2y^2 + 2\right) dy$$



equil. triangles

$$V = \lim_{n\to\infty} \sum \sqrt{3}x^2\, \Delta y$$

$$= \int_{-6}^{1} \frac{\sqrt{3}}{4}\left(\frac{y^3}{3} + 2y^2 + 2\right) dy$$



circles

$$V = \lim_{n\to\infty} \sum \frac{\pi x^2}{2}\, \Delta y$$

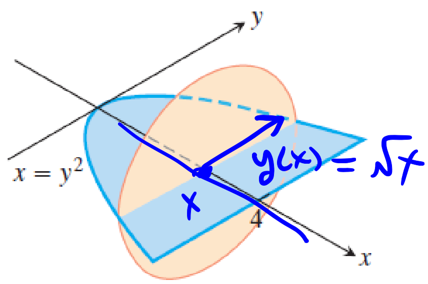
$$= \int_{-6}^{1} \frac{\pi}{8}\left(\frac{y^3}{3} + 2y^2 + 2\right) dy$$

To find the volume of a region, we just need to add up the cross sectional areas:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x = \int_a^b A(x)\,dx$$

The solid lies between the planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run between $x = y^2$.

a. The cross sections are circular discs with diameters in the xy-plane



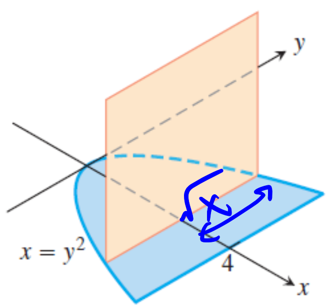
$A(x) = \pi x$

$$V = \lim_{n\to\infty} \sum \pi x\,\Delta x = \int_0^4 \pi x\,dx$$

$$= \frac{\pi x^2}{2}\Big|_0^4$$

$= 8\pi$ cubic units

b. The cross sections are squares with bases in the xy-plane

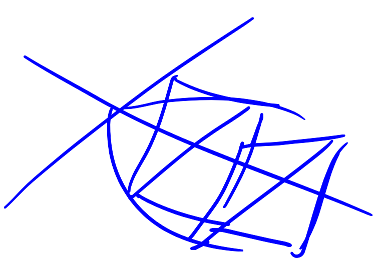


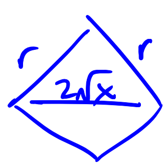
$A(x) = 4x$

$$V = \lim_{n\to\infty} \sum 4x\,\Delta x = \int_0^4 4x\,dx = 2x^2\Big|_0^4$$

$= 32$ cubic units

c. The cross sections are squares with **diagonals** in the xy-plane





$2r^2 = 4x$

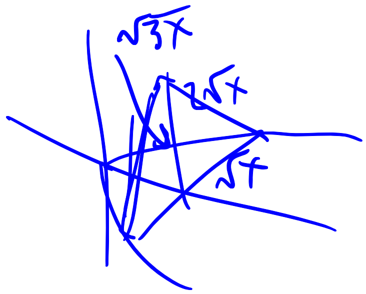
$r^2 = 2x$

$$V = \lim_{n\to\infty} \sum 2x\,\Delta x = \int_0^4 2x\,dx = x^2\Big|_0^4$$

$= 16$

cubic units

d. The cross sections are equilateral triangles with bases in the xy-plane



$A(x) = \sqrt{3}x$

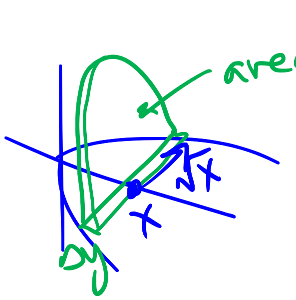
$$V = \lim_{n\to\infty} \sum \sqrt{3}x\,\Delta x = \int_0^4 \sqrt{3}\,x\,dx$$

$$= \frac{\sqrt{3}}{2}x^2\Big|_0^4 = 8\sqrt{3}$$

cubic units

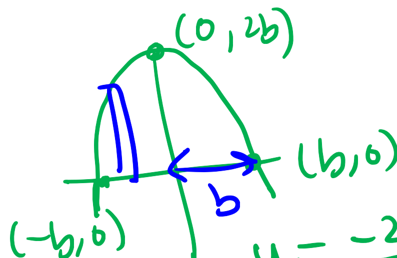
2.718

e. The cross sections are parabolas with bases in the xy-plane (the height and base of the parabolas are the same)



area of parabola

base = height



$$y = \frac{-2}{b}(x-b)(x+b)$$

$$V = \int_0^4 \frac{8}{3}(\sqrt{x})^2\,dx$$

$$= \frac{8}{3}\,\frac{x^2}{2}\Big|_0^4$$

$$= \frac{64}{3}$$

$$\int_{-b}^{b} y\,dx = \int_{-b}^{b} \frac{-2}{b}(x^2-b^2)\,dx$$

$$= \frac{-2}{b}\left[\frac{x^3}{3} - b^2x\right]_{-b}^{b}$$

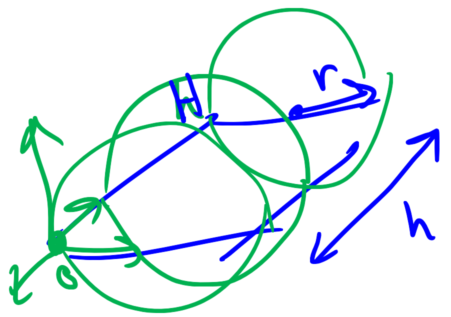
$$= \frac{-4}{b}\left[\frac{b^3}{3} - b^3\right]$$

$$= \frac{8}{3}b^2$$

A fun consequence is that now we can derive equations for volumes of known geometric objects

Example: Show that the volume of a cylinder and cone with radius r and height h is

$$V_1 = \pi r^2 h, \qquad V_2 = \frac{1}{3}\pi r^2 h$$



$$V_1 = \lim_{h \to \infty} \sum \pi r^2 \Delta h$$

$$\int_0^H \pi r^2 \, dh = \pi r^2 H$$

$(0,H)$, $y = \frac{H}{-2r}x + H$

$A = \pi x^2$

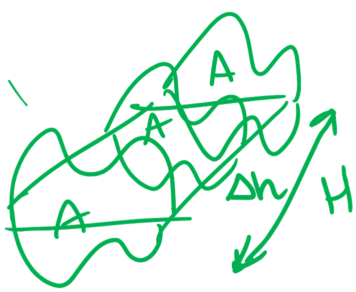
$$\frac{H}{\cancel{2}r} = \frac{H-y}{2x}$$

$$x = \frac{r}{H}(H-y)$$

$$V = \int_0^H \pi \left(\frac{r}{H}(H-y)\right)^2 dy$$

Practice: Show that the volume of a prism with constant cross secrtional area A and height h is just

$$V = Ah$$

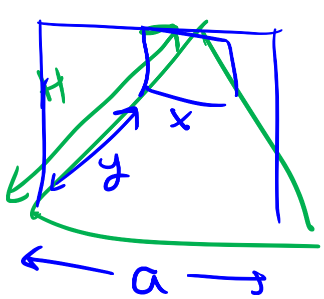


$$V = \lim_{h \to \infty} \sum A \Delta h = \int_0^H A \, dh$$

$$= AH$$

Practice: Show that the volume of a square based pyramid with base side length a and height h is

$$V = \frac{1}{3}a^2 h$$



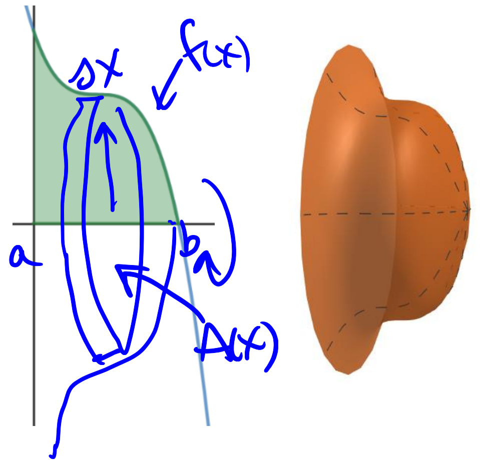
$$\frac{a}{H} = \frac{x}{H-y} \qquad x = \frac{H-y}{H}a$$

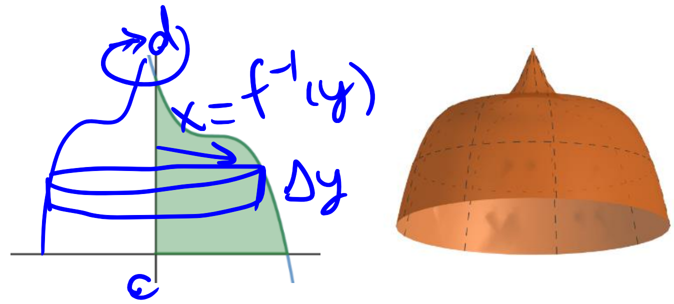
$$V = \sum x^2 \Delta y = \int_0^H a^2 \left(\frac{H-y}{H}\right)^2 dy$$

$$= \frac{a^2}{H^2}\left[\cancel{H^2 y - y^2 H} + \frac{y^3}{3}\right]_0^H$$

$$= \frac{1}{3}a^2 H$$

We are going to finish by briefly talking about how we can generate circular cross sections and that is by simply revolving a region around an axis to generate a volume. Consider the region bound between $x = 0$, $y = 0$ and the curve $y = f(x)$ as shown below





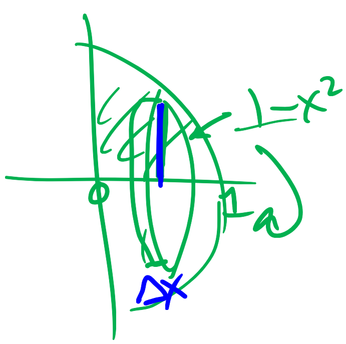
$$V = \int_a^b \pi f^2(x)\,dx$$

No ys

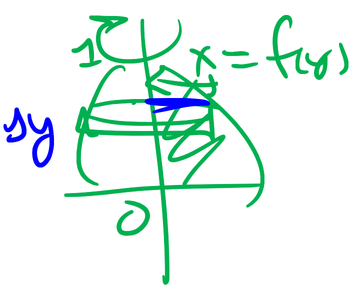
$$V = \int_c^d \pi \left(f^{-1}(y)\right)^2 dy$$

to have ys
No xs

Practice: Find the volume of the solid generated by roating the area between the x and y-axis and the curve $y = 1 - x^2$ if you were to rotate it about the x or y axis.

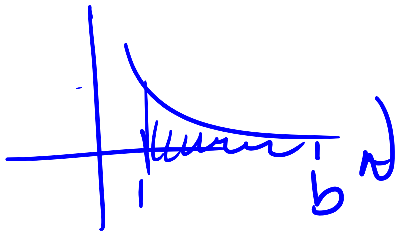


$$V = \int_0^1 \pi(1-x^2)^2\,dx = \int_0^1 \pi(1-2x^2+x^4)\,dx$$
$$= \pi\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1$$
$$= \pi\left(\tfrac{8}{15}\right)$$



$$V = \int_0^1 \pi(1-y)\,dy = \pi\left[y - \tfrac{y^2}{2}\right]_0^1$$
$$= \tfrac{\pi}{2}$$

Practice: Consider the volume formed when the function $f(x) = \frac{1}{x}$ where $x \in [1, b]$ is rotated over the x-axis. What is the volume? What happens when $b \to \infty$?



$$V = \int_1^b \pi f(x)^2 \, dx = \int_1^b \frac{\pi}{x^2} \, dx = -\frac{\pi}{x}\Big|_1^b$$

$$= \pi - \frac{\pi}{b}$$

$$\lim_{b \to \infty} V(b) = \lim_{b \to \infty} \pi - \frac{\pi}{b} = \pi$$

finite volume
infinite length.

Practice Problems: 7.3 # 1-20, 27-38	
Textbook Readings: 7.3 page 383-385	
Workbook Practice: page 328-337	
Next Class: Volumes using washers and shells	