Graphing f from f' and f''

Goal: • Can use the first derivative to identify when the function increases and decreases • Can use the second derivative to identify when the function is concave up and concave down • Can graph a function to have correct extrema and concavity
Terminology: • None

Recall our definitions for the continuous function $f: [a, b] \to \mathbb{R}$
(change inequality direction for decreasing, minimums, and concave down).

Increasing:

- **(First Principles)** If $\forall\, x, y \in [a, b]$ with $x < y$ we get $f(x) < f(y)$, then f is increasing on $[a, b]$
- **(Calculus)** If $f'(x) > 0\ \forall\, x \in (a, b)$ then f is increasing on $[a, b]$

Maximum:

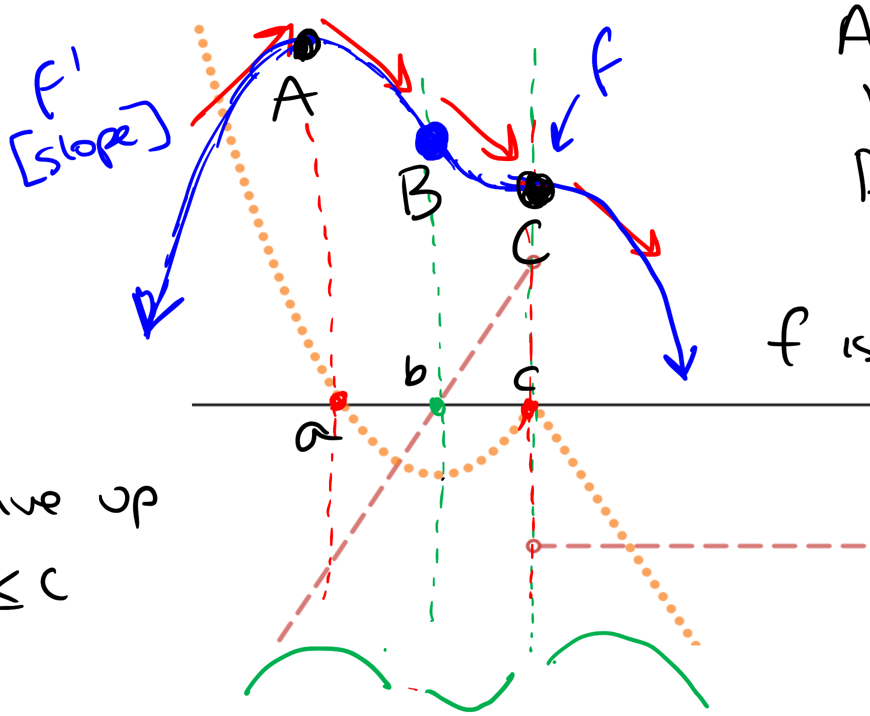
- **(First Principles)** If $f(c) \geq f(x)\ \forall\, x \in [a, b]$, then $x = c$ is an absolute maximum
- **(First Principles)** If $f(c) \geq f(x)$ for every x on some open interval around c, then $x = c$ is a local maximum
- **(Calculus)** If $f'(x) > 0$ when $x \in (a, c)$ and $f'(x) < 0$ when $x \in (c, b)$ then $x = c$ is a local maximum
- **(Calculus)** If $f''(x) < 0$ when $x \in (a, b)$ and $f'(c) = 0$ for $c \in (a, b)$ then $x = c$ is a local maximum

Concavity:

- **(First Principles)** If $f'(x)$ is increasing on $[a, b]$ then it is concave up on $[a, b]$
- **(Calculus)** If $f''(x) > 0\ \forall\, x \in (a, b)$ then f is concave up on $[a, b]$ (assuming $f'(a)$ and $f'(b)$ are definied)

Inflection Points:

- **(Calculus)** If $f''(x)$ is positive or negative when $x \in (a, c)$ and $f''(x)$ is negative or positive when $x \in (c, b)$ then $x = c$ is an inflection point. (f'' needs to change sign)



A is a local max (abs)

B, C are inflection Points

f is increasing on $(-\infty, a]$

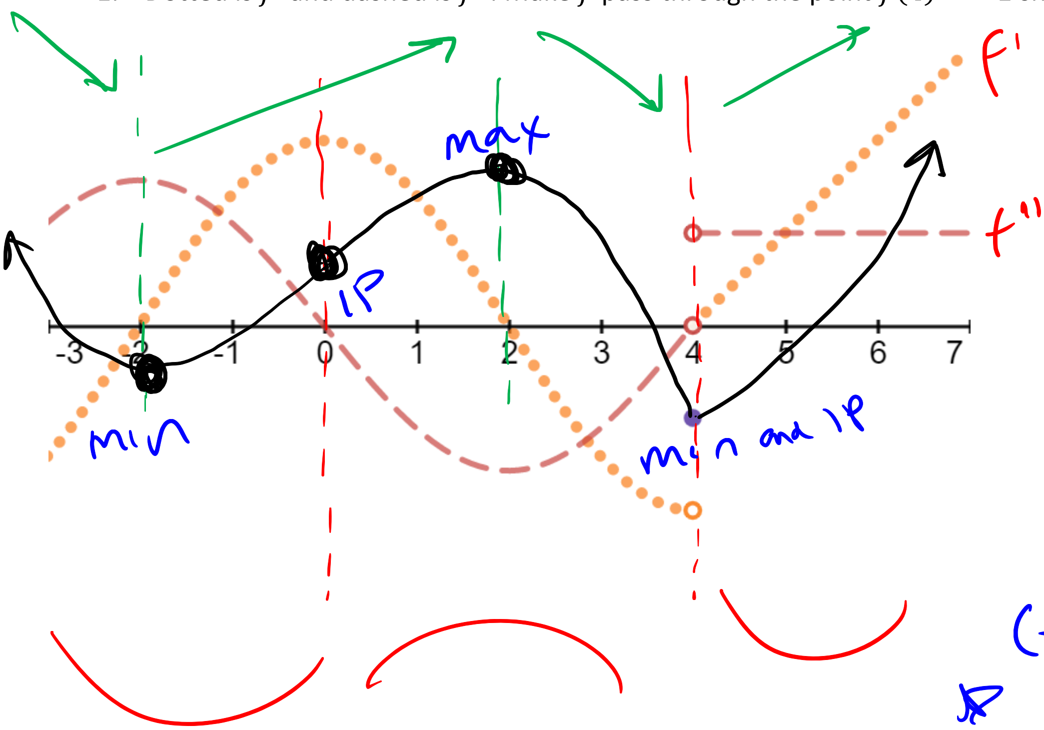
f'' [concavity]

f is concave up $b \leq x \leq c$

Practice:

- Sketch the continuous functions f
- label the max and mins and inflection points
- state the intervals f is increasing and intervals it is concave up.

1. Dotted is f' and dashed is f''. Make f pass through the point $f(4) = -1$ shown below. The domain of f is $\mathbb{R}$

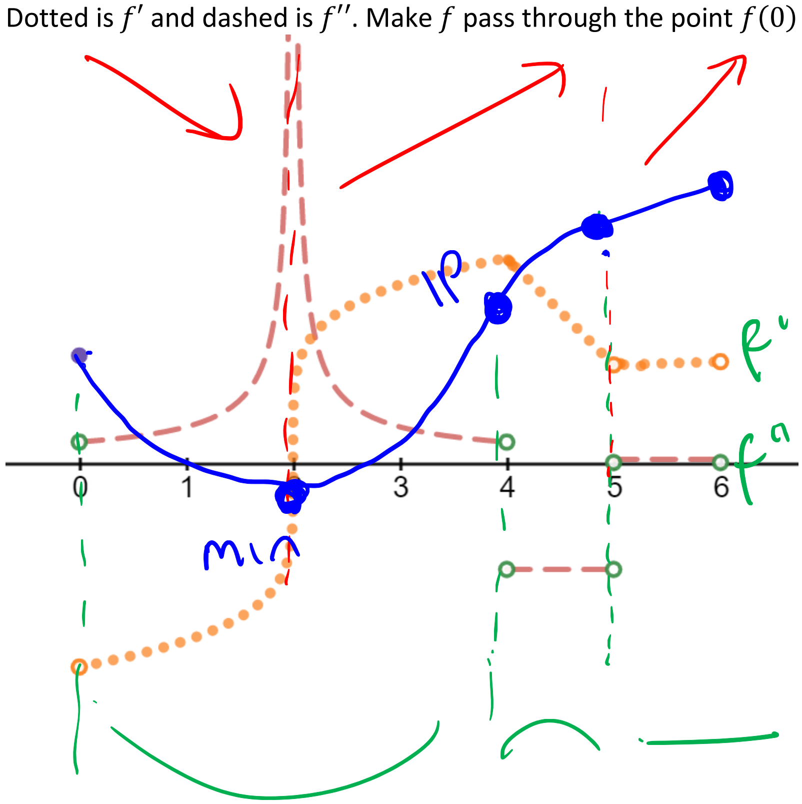


increasing on $[-2, 2] \cup [4, \infty)$

concave up on $(-\infty, 0] \cup (4, \infty)$

* $f'(0) > 0$
  $f'(4) = \emptyset$

2. Dotted is f' and dashed is f''. Make f pass through the point $f(0) = 1$ shown below. The domain of f is $[0, 6]$

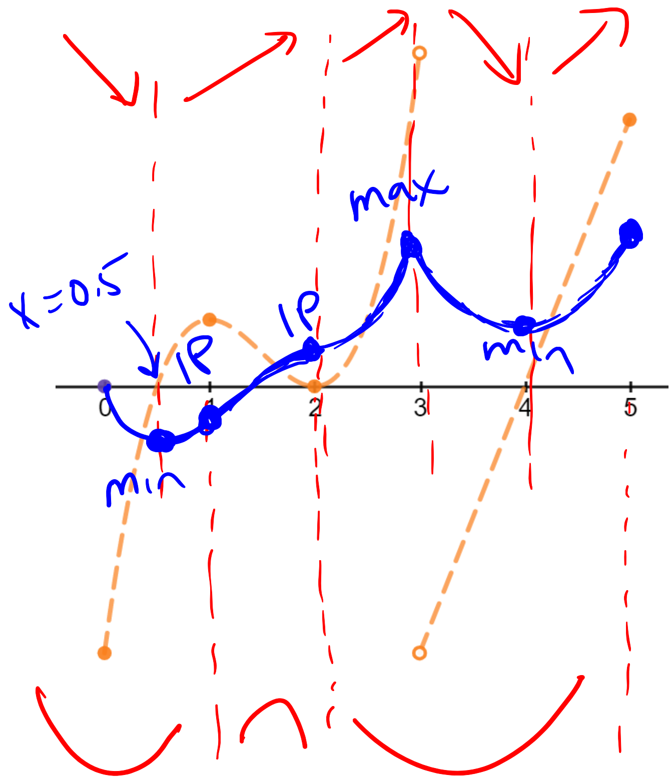


increasing on $[2, \infty)$

concave up on $(0, 4]$

* note that on $[5, 6]$ f is linear

3. Dotted curve is f'. Make f pass through the point $f(0) = 0$ shown below. The domain of f is $[0, 5]$



Increasing on $[0.5, 3] \cup [4, 5]$

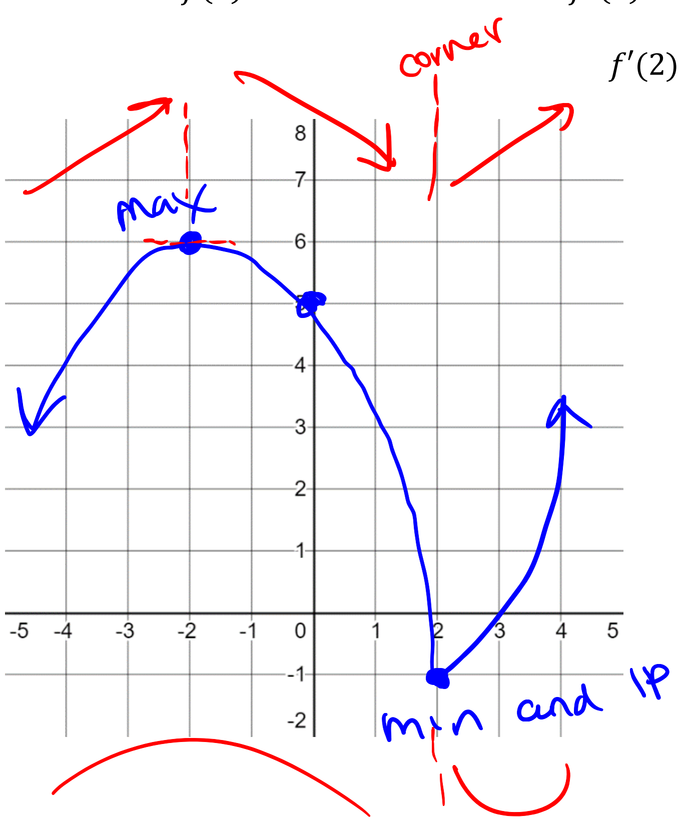
concave up on $[0, 1] \cup [2, 3) \cup (3, 5]$

* I probably should have made $f'(0), f'(5) = \emptyset$

4. The curve has the following properties.

$f(-2) = 6$	$f'(-2) = 0$	$f''(x) > 0$ when $x > 2$
$f(0) = 5$	$f'(x) > 0$ when $\lvert x \rvert > 2$	$f''(x) < 0$ when $x < 2$
$f(2) = -1$	$f'(x) < 0$ when $\lvert x \rvert < 2$	

$f'(2)$ is undefined

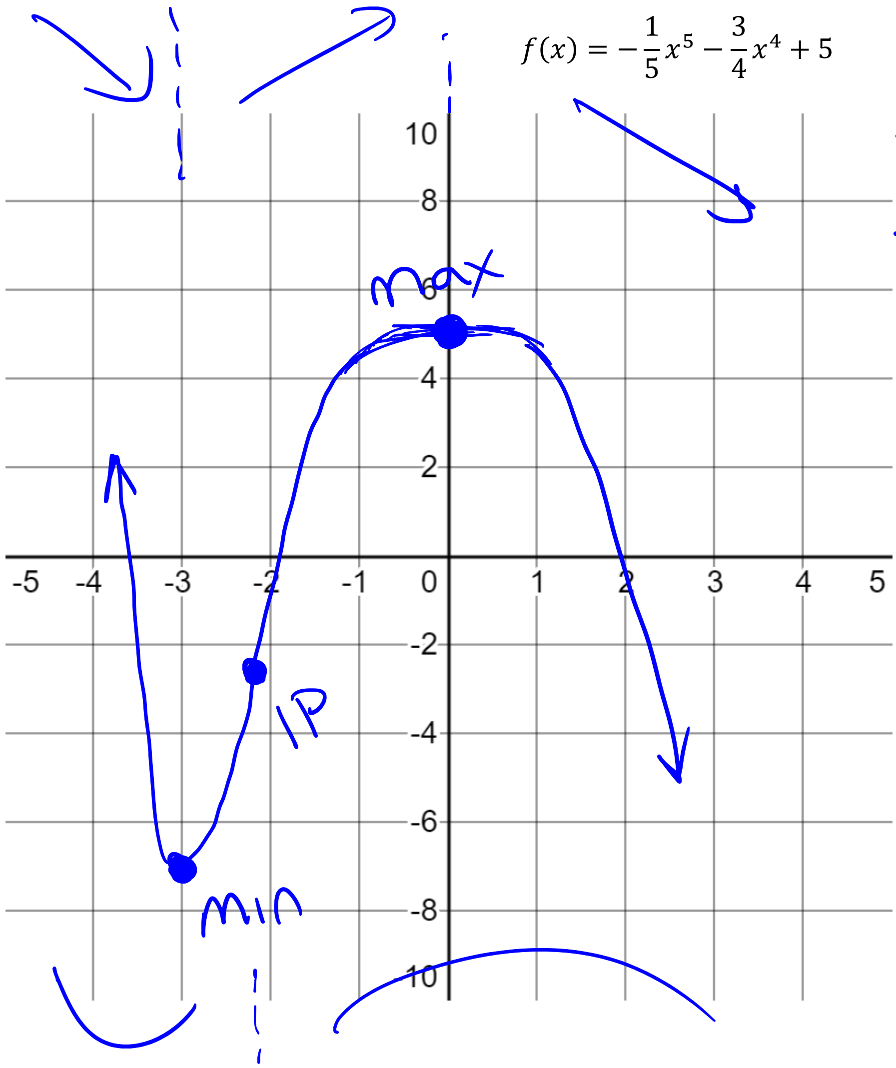


Increasing on $(-\infty, -2] \cup [2, \infty)$

Concave up on $(2, \infty)$ since $f'(2) = \emptyset$

5.

$$f(x) = -\frac{1}{5}x^5 - \frac{3}{4}x^4 + 5$$



$f(0)=5$

$f(-3) = -7.15$

$f(-2.25) = -2.7$

increasing on $[-3, 0]$

concave up on $(-\infty, -\frac{9}{4}]$

$f'(x) = -x^4 - 3x^3$

$= -x^3(x+3)$

$-$ $+$ $-$

-3 0

$f''(x) = -4x^3 - 9x^2$

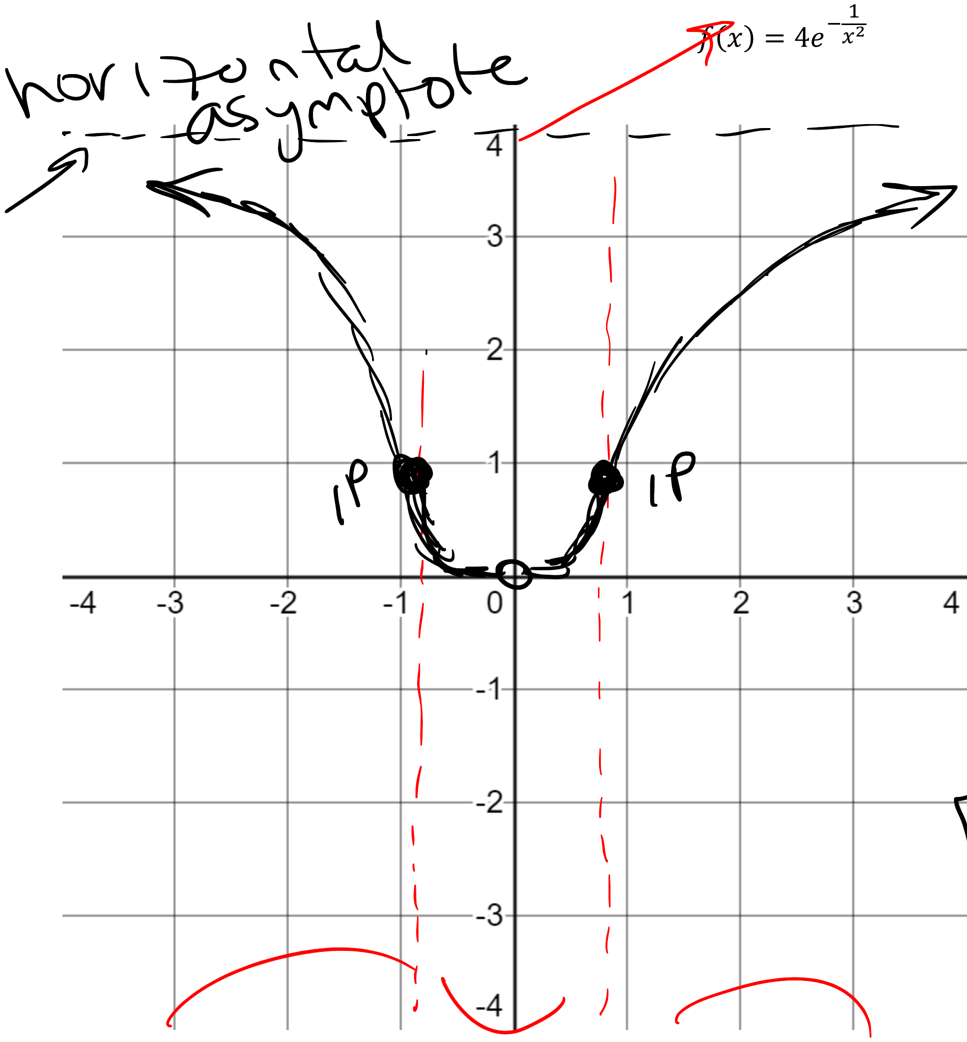
$= -x^2(4x+9)$

$+$ $-$ flat $-$

$-9/4$ 0

6.

$f(x) = 4e^{-\frac{1}{x^2}}$



	$-\sqrt{\frac{2}{3}}$		0		$\sqrt{2/3}$	
f'		$-$	$\emptyset$	$+$		
f''	$-$	0	$+\ \emptyset\ +$		0	$-$

f is increasing when $x>0$

f is concave up on

$\left[-\sqrt{\frac{2}{3}}, 0\right) \cup \left(0, \sqrt{\frac{2}{3}}\right]$

$f(x) = 4e^{-x^{-2}}$ $\lim_{x\to 0} f(x) = 0$

$f'(x) = 4e^{-x^{-2}}(+2x^{-3}) = \dfrac{8e^{-\frac{1}{x^2}}}{x^3}$ CP @ $x=0$

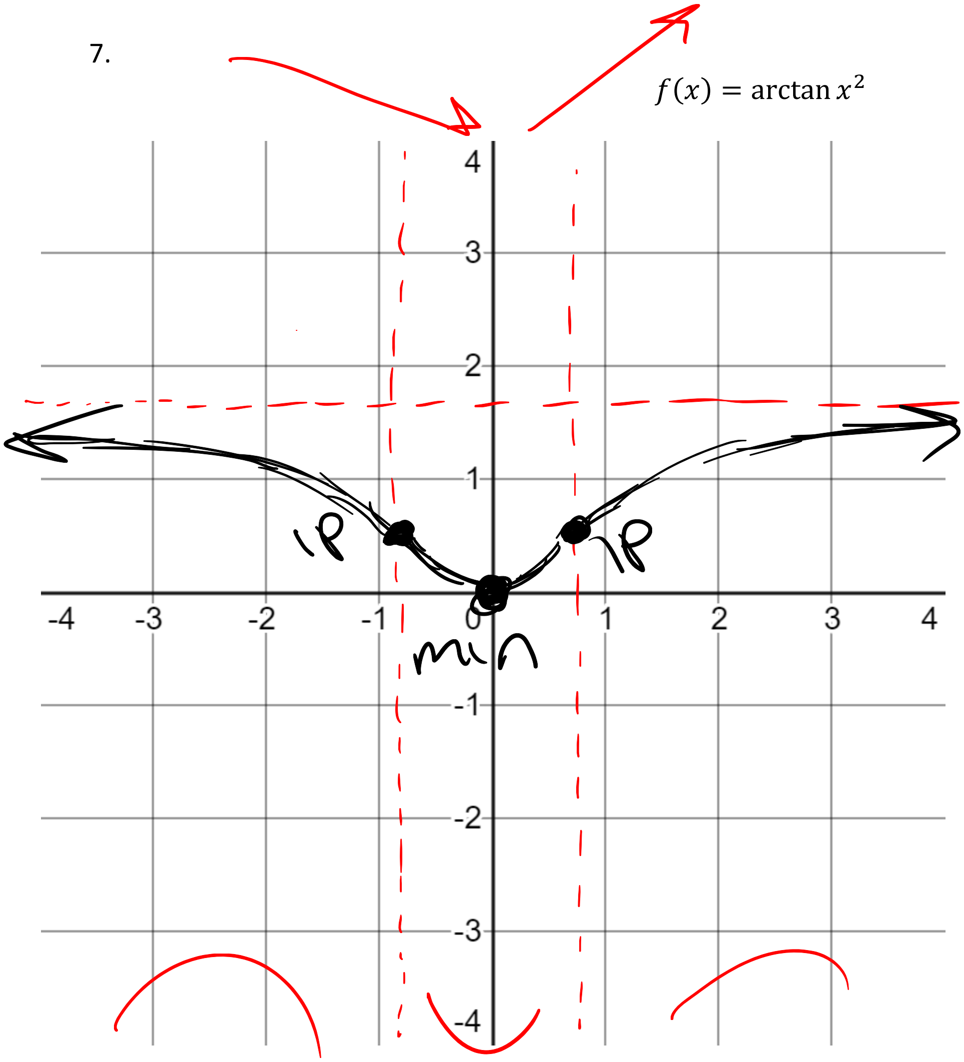
$f''(x) = \left(8e^{-x^{-2}}(2x^{-3})x^3 - 3x^2(8e^{-\frac{1}{x^2}})\right)x^{-6}$

$= 8e^{-x^{-2}}\left[\dfrac{2}{x^6} - \dfrac{3}{x^4}\right] = \dfrac{8e^{-\frac{1}{x^2}}}{x^6}\left[2 - 3x^2\right]$

$f''(x) = 0$ when $x = \pm\sqrt{\frac{2}{3}}$ $f''(x) = \emptyset$ when $x=0$

7.

$f(x) = \arctan x^2$



$-\sqrt[4]{\frac{1}{3}}$ $\quad 0 \quad$ $\sqrt[4]{1/3}$

f' $\quad - \quad 0 \quad +$

f'' $\quad - \quad 0 \quad + \quad 0 \quad -$

$y = \frac{\pi}{2}$ asymptote

f is increasing when $x \geq 0$

f is concave up on $\left[-\sqrt[4]{\frac{1}{3}}, \sqrt[4]{\frac{1}{3}}\right]$

$$f'(x) = \frac{1}{x^4+1} \cdot 2x \qquad x=0 \text{ is CP}$$

$$f''(x) = \frac{2(x^4+1) - 4x^3(2x)}{(x^4+1)^2} = \frac{2-6x^4}{(x^4+1)^2}$$

$f''(x) = 0$ when $x = \pm\sqrt[4]{\frac{1}{3}}$

Practice Problems: 4.3: # 31-36, 45-48
Textbook Readings: 4.3 page 201-202
Workbook Practice: page 194-197, 200-203
Next Day: Optimization