Log Laws

Goal:
- Can derive the product law using exponent laws
- Can derive the power law and quotient law from the product law.
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Terminology:
- Product Law
- Quotient Law
- Power Law
- Change of Base Law

In grade 9 and 10 you learned about the exponent laws and know that

$$b^n \cdot b^m = b^{n+m} \qquad (b^n)^m = b^{nm} \qquad b^{-1} = \frac{1}{b}$$

$$\underbrace{b \cdot b \cdots b}_{n} \cdot \underbrace{b \cdot b \cdots b}_{m} = \underbrace{b \cdot b \cdots b}_{n+m} \qquad \underbrace{b^n \cdot b^n \cdots b^n}_{m} = b^{\overbrace{n+\cdots+n}^{m}} \qquad \text{By definition}$$

Using function notation if $g(x) = b^x$ then the above laws for exponents give defining characteristics

$$g(n) \cdot g(m) = g(n+m) \qquad g(n)^m = g(nm) \qquad g(-x) = \frac{1}{g(x)}$$

Logarithms, being the inverse of exponentials have similar laws:

$f(x) = \log_b x$

$f(m \cdot n) = f(m) + f(n)$

Product Law: $\log_b(m \cdot n) = \log_b m + \log_b n$

Power Law: $\log_b x^n = n \cdot \log_b x$

Quotient Law: $\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$

→ logs turn products into sums of logs

Product Law Proof:

$\underbrace{\log_b(m \cdot n)}_{=z} = \underbrace{\log_b m}_{=x} + \underbrace{\log_b n}_{=y}$

WTS
want to show $\boxed{z = x+y}$

$\cancel{b^{\log_b}} m \cdot n = b^z$ $\qquad \cancel{b^{\log_b}} m = b^x$ $\qquad \cancel{b^{\log_b}} n = b^y$

$\boxed{m \cdot n} = b^z$ $\qquad m = b^x$ $\qquad n = b^y$

$\Rightarrow b^x \cdot b^y = b^z = b^{x+y} \Rightarrow z = x+y$ done.

Power Law Proof:

exponent product

$$\log_b x^n = n \cdot \log_b x$$

$$\log_b(\underbrace{x \cdot x \cdots x}_{n \text{ times}}) = \underbrace{\log_b x + \log_b x + \cdots + \log_b x}_{n \text{ times}}$$

$$= n \log_b x$$

Quotient Law Proof:

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b(m \cdot n^{-1}) = \log_b m + \log_b n^{-1}$$

$$= \log_b m - \log_b n \quad \text{done}$$

Practice: Use log laws to simplify the following as a single log:

$(\log_2 7) - (2\log_2 3) + (\log_2 6)$

$\log_2 7 - \log_2 3^2 + \log_2 6$

$\log_2\left(\frac{7}{3^2}\right) + \log_2 6$

$\log_2\left(\frac{7 \cdot \cancel{6}^{2}}{\cancel{9}_{3}}\right) = \log_2\left(\frac{14}{3}\right)$

$3\ln 6 - \ln 9 - \ln 8$

$\ln 3$

$2\log(12+3) - (\log 5 + \log 4)$

$\log(45/4)$

$\log e - \ln 10$

$10^{\log e - \ln 10} = 10^x$

$e^{\cancel{\log e}} \cdot 10^{-\ln 10} = 10^x$

not the same base

$\log x$

$x > 0$

We need to be careful about the domain when we simplify log functions:

Example: Simplify the following and state the overall domain.

$$f(x) = -\log_6 \sqrt{x} + 4.5 \log_6 x$$

$x > 0$ $\boxed{x > 0}$

$= 4.5 \log_6 x - \log_6 \sqrt{x}$

$= \log_6 x^{4.5} - \log_6 x^{1/2} = \log_6 \frac{x^{4.5}}{x^{0.5}}$

$\boxed{= \log_6 x^4}$

Practice: Simplify the following and state the overall domain

$$g(x) = \log x + 2\log(x+1) - \log((x+1)(x-2))$$

$x > 0$ $x > -1$ $x > 2$ or $x < -1$

-1 2

$g(x) = \log\left(\frac{x(x+1)^2}{(x+1)(x-2)}\right)$

$= \log\left(\frac{x(x+1)}{x-2}\right)$

$\boxed{x > 2}$

Aside from simplifying, expanding can help us simply an expression if we can evaluate logs of the smaller pieces using:

Inverse Property: $\log_b b^x = x$ and $b^{\log_b x} = x$

Practice: Given that $\log_2 5 = A$ and $\log_2 3 = B$ simplify the following.

$\log_2 40\sqrt{5}$ $8 \cdot 5 \cdot 5^{1/2}$

$\log_2 2^3 \cdot 5^{3/2}$

$\log_2 2^3 + \log_2 5^{3/2}$

$3 + \frac{3}{2}\log_2 5$

$3 + \frac{3}{2}A$

$\log_2 \left(\frac{36}{5}\right)^3$

$\log_2 \left(\frac{2^2 \cdot 3^2}{5}\right)^3$

$\log_2 \frac{2^6 \cdot 3^6}{5^3}$

$\log_2 2^6 + \log_2 3^6 - \log_2 5^3$

$6 + 6B - 3A$

$\log_7 100 = \frac{\log 100}{\log 7}$ $\frac{\log x}{\log 7} = \log_7 x = \frac{\ln x}{\ln 7}$

Change of Base Law: $\log_b a = \frac{\log_x a}{\log_x b} = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$

Proof:

$b^{\boxed{\log_b a = y}}$ $\log_x a = \log_x b^y$

$\log_x (a = b^y)$ $\log_x a = y \cdot \log_x b$

$\boxed{\frac{\log_x a}{\log_x b} = y}$

Practice: Transform the following functions to the common log and the natural log.

$f(x) = \log_2 x = \frac{\log x}{\log 2} = 3.3 \log x$

$= \frac{\ln x}{\ln 2} = 1.44 \ln x$ $= \frac{1}{\log 2} \cdot \log x$

$g(x) = -2 \log_{100} x$

$= -2 \frac{\log x}{\log 100} = -2 \frac{\ln x}{\ln 100}$

$= - \log x$

$= -0.43 \ln x$

$h(x) = \log_{e^3}(x-1) + 2$

$\frac{\log(x-1)}{\log e^3} + 2 = \frac{\ln(x-1)}{\ln e^3} + 2$

$= 0.77 \log(x-1) + 2$

$= \frac{1}{3} \ln(x-1) + 2$

$k(x) = \log_{10^n} x$

$\frac{\log x}{\log 10^n} = \frac{\ln x}{\ln 10^n}$

$= \frac{1}{n} \log x$

$= \frac{0.43}{n} \ln x$

Finally, a word about how **log scales** alter perspective. When things grow logarithmically (very slowly) we can change our base to whatever we want (common or natural). In this way, we will get some function

$$f(x) \propto \log x$$

And if $\Delta f = f(x_2) - f(x_1) = n$, then that means $n \propto \log x_2 - \log x_1 = \log\left(\frac{x_2}{x_1}\right)$ so $\frac{x_2}{x_1} \propto 10^n$

Example: There are a lot of ways to measure earthquake sizes, but the most common measure is the Moment Magnitude which describes the amount of energy released according to the formula:

$$M = \frac{2}{3}\log E + 1.22$$

Where E is the energy released as kg of TNT.

a. The 2011 earthquake in Japan had a moment magnitude of 9.1. Compare that to the earthquake in Vancouver in 2015 which had a magnitude of 4.8. How much stronger was the earthquake in Japan?

b. If the magnitude increases by 1 how much will the energy increase by?

comparing growth rates

Suggested Practice Problems: 8.3 page 400-403 # 1-3, 5, 6, 8-13, 17-20, C1	
Textbook Reading: 8.3 page 392-399 Key Ideas on page 400	
Next Class: Practice solving and modelling log and exponential equations	

Exponential Growth

Log Laws