

Log Laws

<p>Goal:</p> <ul style="list-style-type: none"> • Can derive the product law using exponent laws • Can derive the power law and quotient law from the product law. •
<p>Terminology:</p> <ul style="list-style-type: none"> • Product Law • Quotient Law • Power Law • Change of Base Law

In grade 9 and 10 you learned about the exponent laws and know that

$$b^n \cdot b^m = b^{n+m} \qquad (b^n)^m = b^{nm} \qquad b^{-1} = \frac{1}{b}$$

$$\underbrace{b \cdot b \cdots b}_n \cdot \underbrace{b \cdot b \cdots b}_m = \underbrace{b \cdot b \cdots b}_{n+m} \qquad \underbrace{b^n \cdot b^n \cdots b^n}_m = b^{\frac{n+\cdots+n}{m}}$$

By definition

Using function notation if $g(x) = b^x$ then the above laws for exponents give defining characteristics

$$g(n) \cdot g(m) = g(n+m) \qquad g(n)^m = g(nm) \qquad g(-x) = \frac{1}{g(x)}$$

Logarithms, being the inverse of exponentials have similar laws:

$f(x) = \log_b x$

$f(m \cdot n) = f(m) + f(n)$

<p>Product Law: $\log_b(m \cdot n) = \log_b m + \log_b n$</p> <p>Power Law: $\log_b x^n = n \cdot \log_b x$</p> <p>Quotient Law: $\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$</p>
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Product Law Proof:

$$\log_b(m \cdot n) = \log_b m + \log_b n$$

logs turn products into sums of logs

WTS want to show $z = x + y$

$$b^{\log_b m \cdot n} = b^z \qquad b^{\log_b m} = b^x \qquad b^{\log_b n} = b^y$$

$$m \cdot n = b^z \qquad m = b^x \qquad n = b^y$$

$$\Rightarrow b^x \cdot b^y = b^z = b^{x+y} \Rightarrow z = x + y \text{ done.}$$

Power Law Proof:

$$\log_b x^n = n \cdot \log_b x$$

exponent
product

$$\log_b (\underbrace{x \cdot x \cdots x}_{n \text{ times}}) = \log_b x + \log_b x + \cdots + \log_b x$$

n times
n times

$$= n \log_b x$$

Quotient Law Proof:

$$\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b (m \cdot n^{-1}) = \log_b m + \log_b n^{-1}$$

done

Practice: Use log laws to simplify the following as a single log:

$$(\log_2 7) - 2 \log_2 3 + \log_2 6$$

$$3 \ln 6 - \ln 9 - \ln 8$$

$$\log_2 7 - \log_2 3^2 + \log_2 6$$

$$\log_2 \left(\frac{7}{3^2}\right) + \log_2 6$$

$$\log_2 \left(\frac{7 \cdot 6}{9 \cdot 3}\right) = \log_2 \left(\frac{14}{3}\right)$$

$$\ln 3$$

$$2 \log(12 + 3) - (\log 5 + \log 4)$$

$$10^{\log e - \ln 10} = 10^x$$

$$e^{\log e} \cdot 10^{-\ln 10} = 10^x$$

not the same base

$$\log \left(\frac{45}{4}\right)$$



We need to be careful about the domain when we simplify log functions:

Example: Simplify the following and state the overall domain.

$$f(x) = -\log_6 \sqrt{x} + 4.5 \log_6 x \quad x > 0$$

$$= 4.5 \log_6 x - \log_6 \sqrt{x}$$

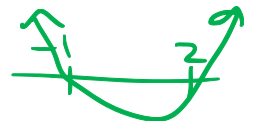
$$= \log_6 x^{4.5} - \log_6 x^{1/2} = \log_6 \frac{x^{4.5}}{x^{0.5}}$$

$$= \log_6 x^4$$

Practice: Simplify the following and state the overall domain

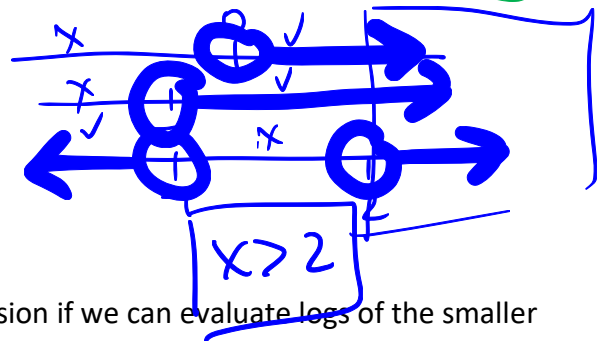
$$g(x) = \log(x) + 2 \log(x+1) - \log((x+1)(x-2))$$

$x > 0$ $x > -1$ $x > 2$ or $x < -1$



$$g(x) = \log \left(\frac{x(x+1)^2}{(x+1)(x-2)} \right)$$

$$= \log \left(\frac{x(x+1)}{x-2} \right)$$



Aside from simplifying, expanding can help us simply an expression if we can evaluate logs of the smaller pieces using:

Inverse Property: $\log_b b^x = x$ and $b^{\log_b x} = x$

Practice: Given that $\log_2 5 = A$ and $\log_2 3 = B$ simplify the following.

$$\log_2 40 \sqrt{5} \quad 8 \cdot 5 \cdot 5^{1/2}$$

$$\log_2 \left(\frac{36}{5} \right)^3$$

$$\log_2 2^3 \cdot 5^{3/2}$$

$$\log_2 2^3 + \log_2 5^{3/2}$$

$$3 + \frac{3}{2} \log_2 5$$

$$3 + \frac{3}{2} A$$

$$\log_2 \left(\frac{2^2 \cdot 3^2}{5} \right)^3$$

$$\log_2 \frac{2^6 \cdot 3^6}{5^3}$$

$$\log_2 2^6 + \log_2 3^6 - \log_2 5^3$$

$$6 + 6B - 3A$$

$$\log_7 100 = \frac{\log 100}{\log 7} \quad \frac{\log X}{\log 7} = \log_7 X = \frac{\ln X}{\ln 7}$$

$$\text{Change of Base Law: } \log_b a = \frac{\log_x a}{\log_x b} = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$$

Proof:

$$b^{\log_b a = y} \quad \log_x (a = b^y)$$

$$\log_x a = \log_x b^y$$

$$\log_x a = y \cdot \log_x b$$

$$\frac{\log_x a}{\log_x b} = y$$

Practice: Transform the following functions to the common log and the natural log.

$$f(x) = \log_2 x = \frac{\log x}{\log 2} = 3.3 \log x$$

$$= \frac{\ln x}{\ln 2} = 1.44 \ln x = \frac{1}{\log 2} \cdot \log x$$

$$g(x) = -2 \log_{100} x$$

$$= -2 \frac{\log x}{\log 100} = -2 \frac{\ln x}{\ln 100}$$

$$= -\log x$$

$$= -0.43 \ln x$$

$$h(x) = \log_{e^3} (x-1) + 2$$

$$\frac{\log(x-1)}{\log e^3} + 2 = \frac{\ln(x-1)}{\ln e^3} + 2$$

$$= 0.77 \log(x-1) + 2$$

$$= \frac{1}{3} \ln(x-1) + 2$$

$$k(x) = \log_{10^n} x$$

$$\frac{\log x}{\log 10^n} = \frac{\ln x}{\ln 10^n}$$

$$= \frac{1}{n} \log x$$

$$= \frac{0.43}{n} \ln x$$

Finally, a word about how **log scales** alter perspective. When things grow logarithmically (very slowly) we can change our base to whatever we want (common or natural). In this way, we will get some function

$$f(x) \propto \log x$$

And if $\Delta f = f(x_2) - f(x_1) = n$, then that means $n \propto \log x_2 - \log x_1 = \log \left(\frac{x_2}{x_1} \right)$ so $\frac{x_2}{x_1} \propto 10^n$

Example: There are a lot of ways to measure earthquake sizes, but the most common measure is the Moment Magnitude which describes the amount of energy released according to the formula:

$$M = \frac{2}{3} \log E + 1.22$$

Where E is the energy released as kg of TNT.

- a. The 2011 earthquake in Japan had a moment magnitude of 9.1. Compare that to the earthquake in Vancouver in 2015 which had a magnitude of 4.8. How much stronger was the earthquake in Japan?

- b. If the magnitude increases by 1 how much will the energy increase by?

comparing growth rates
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Suggested Practice Problems: 8.3 page 400-403 # 1-3, 5, 6, 8-13, 17-20, C1

Textbook Reading: 8.3 page 392-399

Key Ideas on page 400

Next Class: Practice solving and modelling log and exponential equations

