Log Laws

Goal:
- Can derive the product law using exponent laws
- Can derive the power law and quotient law from the product law.
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Terminology:
- Product Law
- Quotient Law
- Power Law
- Change of Base Law

In grade 9 and 10 you learned about the exponent laws and know that

$$b^n \cdot b^m = b^{n+m} \qquad (b^n)^m = b^{nm} \qquad b^{-1} = \frac{1}{b}$$

$$\underbrace{b \cdot b \cdots b}_{n} \cdot \underbrace{b \cdot b \cdots b}_{m} = \underbrace{b \cdot b \cdots b}_{n+m} \qquad \underbrace{b^n \cdot b^n \cdots b^n}_{m} = b^{\overbrace{n+\cdots+n}^{m}} \qquad \text{By definition}$$

Using function notation if $g(x) = b^x$ then the above laws for exponents give defining characteristics

$$g(n) \cdot g(m) = g(n+m) \qquad g(n)^m = g(nm) \qquad g(-x) = \frac{1}{g(x)}$$

Logarithms, being the inverse of exponentials have similar laws:

Product Law: $\log_b(m \cdot n) = \log_b m + \log_b n$

Power Law: $\log_b x^n = n \cdot \log_b x$

Quotient Law: $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$

$f(x) = \log_b x$

$f(m \cdot n) = f(m) + f(n)$

Inner product   outer sum

Product Law Proof:

$$\underbrace{\log_b(m \cdot n)}_{A} = \underbrace{\log_b m}_{B} + \underbrace{\log_b n}_{C}$$

WTS: $A = B + C$
want to show

$\cancel{b^{\log_b}}(mn) = b^A \Rightarrow m \cdot n = \boxed{b^A = b^B \cdot b^C} = b^{B+C}$

$\cancel{b^{\log_b}} m = b^B \Rightarrow m = b^B$

same

$n = b^C$

$\Downarrow$

$A = B + C$

done

Power Law Proof: $\log_b (x^n) = n \cdot \log_b x$

$$\log_b(\underbrace{x \cdot x \cdots x}_{n}) = \underbrace{\log_b x + \log_b x + \cdots + \log_b x}_{n}$$

$$= n \cdot \log_b x \quad \text{done}$$

Quotient Law Proof:

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b(m \cdot n^{-1}) = \log_b m + -1\log_b n$$

$$= \log_b m - \log_b n \quad \text{done}$$

Practice: Use log laws to simplify the following as a single log:

$\log_2 7 - 2\log_2 3 + \log_2 6$

$$\underbrace{\log_2 7 - \log_2 9}_{} + \log_2 6$$

$$\log_2\left(\frac{7}{9}\right) + \log_2 6$$

$$\log_2\left(\frac{7 \cdot \cancel{6}^{2}}{\cancel{9}_{3}}\right) = \log_2\left(\frac{14}{3}\right)$$

$3\ln 6 - \ln 9 - \ln 8$

$$\ln 6^3 - \ln 9 - \ln 8$$

$$\ln\left(\frac{6^3}{9 \cdot 8}\right) = \ln 3$$

$2\log(12+3) - (\log 5 + \log 4)$

$$2\log 15 - \log 20$$

$$\log 15^2 - \log 20$$

$$\log(15^2/20) = \log \frac{45}{4}$$

$\log e - \ln 10$

$$e^x \cdot 10^y = :($$

NOT The same base.

Domain of $\log x$, $x>0$

We need to be careful about the domain when we simplify log functions:

Example: Simplify the following and state the overall domain.

$$f(x) = -\log_6 \sqrt{x} + 4.5 \log_6 x$$

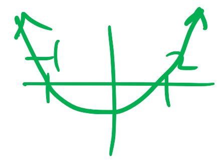
$\sqrt{x}>0 \Rightarrow x>0$ $\boxed{x>0}$ Domain

$$f(x) = 4.5\log_6 x - \log_6 x^{1/2}$$
$$= \log_6 x^{4.5} - \log_6 x^{0.5}$$
$$= \log_6 \left(x^{4.5}/x^{0.5} \right) = \log_6 x^4 \quad , x>0$$

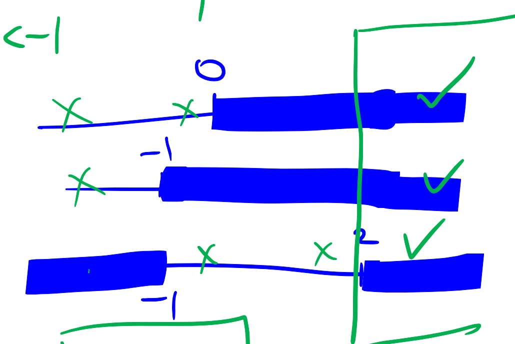
Practice: Simplify the following and state the overall domain

$$g(x) = \log x + 2\log(x+1) - \log\big((x+1)(x-2)\big)$$

$x>0$ $\quad x+1>0$, $x>-1$ $\quad x>2$ OR $x<-1$



$$g(x) = \log x + \log(x+1)^2 - \log\,(x+1)(x-2)$$
$$= \log\left(\frac{x(x+1)^{\cancel{2}}}{\cancel{(x+1)}(x-2)}\right) = \log\left(\frac{x(x+1)}{x-2}\right)$$



$\boxed{x>2}$

Aside from simplifying, expanding can help us simply an expression if we can evaluate logs of the smaller pieces using:

Inverse Property: $\log_b b^x = x$ and $b^{\log_b x} = x$

Practice: Given that $\boxed{\log_2 5 = A}$ and $\boxed{\log_2 3 = B}$ simplify the following.

$$\log_2 40\sqrt{5}$$

$$\log_2 (2^3 \cdot 5 \cdot 5^{1/2})$$
$$\cancel{\log_2 2^3} + \log_2 5^{1.5}$$
$$3 + 1.5\log_2 5$$
$$\boxed{3 + 1.5A}$$

$$\log_2 \left(\frac{36}{5}\right)^3$$

$$\log_2\left(\frac{2^2\cdot 3^2}{5}\right)^3$$
$$\log_2\left(\frac{2^6\cdot 3^6}{5^3}\right)$$
$$\cancel{\log_2 2^6} + \log_2 3^6 - \log_2 5^3$$
$$6 + 6\log_2 3 - 3\log_2 5$$

$6+6B-3A$

Change of Base Law: $\log_b a = \frac{\log_x a}{\log_x b} = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$

ugly ↑ nice

Proof:

$b^{\log_b a} = b^N \Rightarrow \log_x (a = b^N)$

$\log_x a = \log_x b^N$

$\log_x a = N \log_x b$

$\frac{\log_x a}{\log_x b} = N = \log_b a$

$f(x) = \log_3 x$

$f(8)$??

$\log_3 x = \frac{\log x}{\log 3} = \frac{\ln x}{\ln 3}$

$\boxed{= 2.1 \log x = 0.9 \ln x}$

$f(8) = 2.1 \log 8 = 0.9 \ln 8$

Practice: Transform the following functions to the common log and the natural log.

$f(x) = \log_2 x$

$= \frac{\log x}{\log 2} = \boxed{\frac{1}{\log 2}} \cdot \log x$

$g(x) = -2\log_{100} x$

$h(x) = \log_{e^3}(x-1) + 2$

$\frac{\log(x-1)}{\log e^3} + 2 = 0.77 \log(x-1) + 2$

$\frac{\ln(x-1)}{\cancel{\ln e^3}} + 2 = \frac{1}{3}\ln(x-1) + 2$

$k(x) = \log_{10^n} x$

$\frac{\log x}{\log 10^n} = \frac{\log x}{n}$

$\frac{\ln x}{\ln 10^n} = \frac{\ln x}{n \cdot 2.3} = \frac{0.43}{n} \ln x$

Finally, a word about how **log scales** alter perspective. When things grow logarithmically (very slowly) we can change our base to whatever we want (common or natural). In this way, we will get some function

$$f(x) \propto \log x$$

And if $\Delta f = f(x_2) - f(x_1) = n$, then that means $n \propto \log x_2 - \log x_1 = \log\left(\frac{x_2}{x_1}\right)$ so $\frac{x_2}{x_1} \propto 10^n$

Example: There are a lot of ways to measure earthquake sizes, but the most common measure is the Moment Magnitude which describes the amount of energy released according to the formula:

$$M = \frac{2}{3}\log E + 1.22$$

Where E is the energy released as kg of TNT.

a. The 2011 earthquake in Japan had a moment magnitude of 9.1. Compare that to the earthquake in Vancouver in 2015 which had a magnitude of 4.8. How much stronger was the earthquake in Japan?

b. If the magnitude increases by 1 how much will the energy increase by?

Suggested Practice Problems: 8.3 page 400-403 # 1-3, 5, 6, 8-13, 17-20, C1
Textbook Reading: 8.3 page 392-399 Key Ideas on page 400
Next Class: Practice solving and modelling log and exponential equations

Exponential Growth

Log Laws