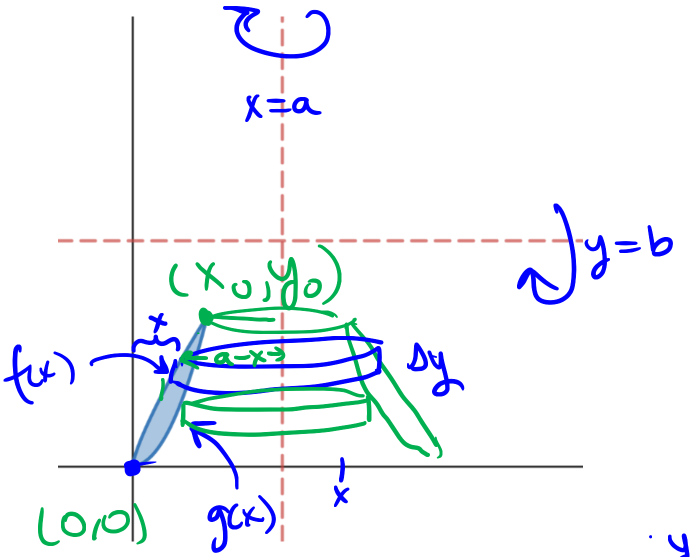
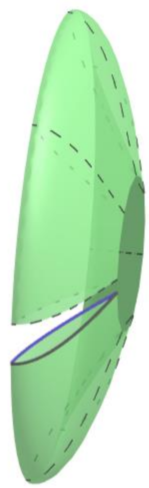
Volumes Part 2: Washers and Shells

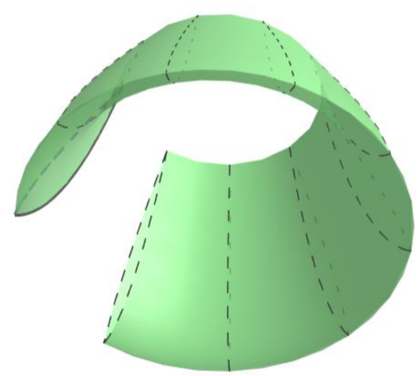
Goal:
- Can determine the volume of solids after rotation around any line.
- Can use washers and discs to find the volume of a shape.
- Can determine the volume of solids after rotation using shells

Terminology:
- None

Volume using washers:
The region bound between the two one-to-one functions $y = f(x)$ and $y = g(x)$ is rotated around
a. The line $x = a$
b. The line $y = b$
Write the expression for the volume formed.The curves intersect at the origin and the point (x_0, y_0) and $\forall\, x \in (0, x_0)$

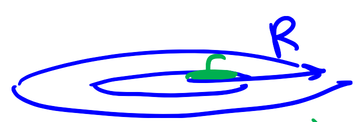






$$V = \int_0^{y_0} \pi (a - x_f)^2\, dy - \int_0^{y_0} \pi (a - x_g)^2\, dy$$

$$= \pi \int_0^{y_0} \left[(a - x_f)^2 - (a - x_g)^2\right] dy$$

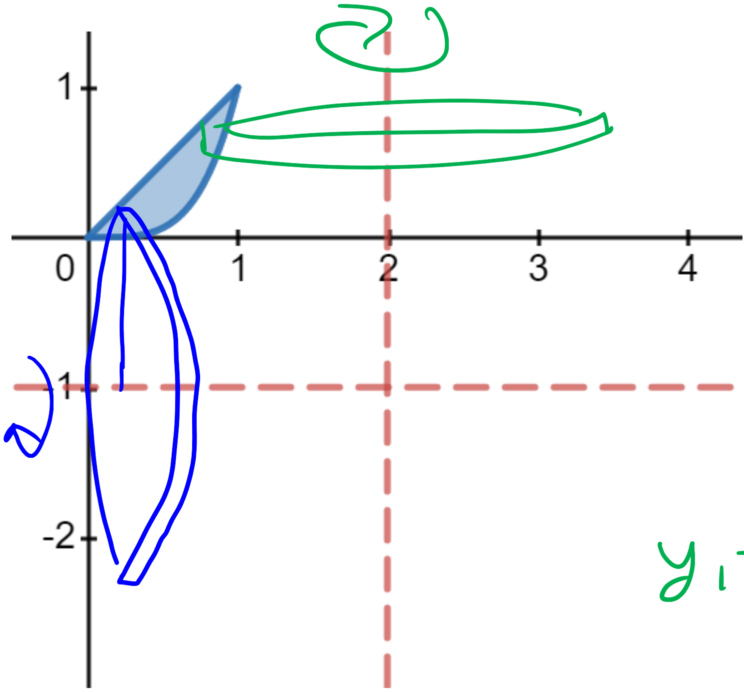


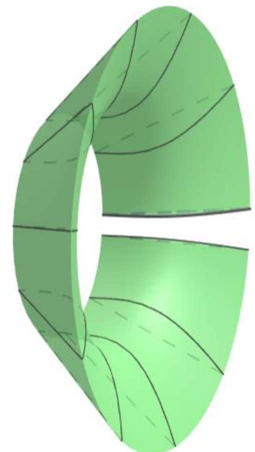
$A = \pi (R^2 - r^2)$

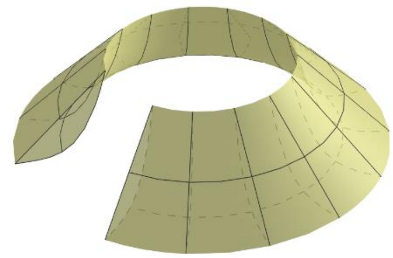
$$= \pi \int_0^{y_0} \left[(a - f^{-1}(y))^2 - (a - g^{-1}(y))^2\right] dy$$ about $x = a$

$$V = \int_0^{x_0} \pi \left[(b - g(x))^2 - (b - f(x))^2\right] dx$$

Practice: The region bound between the curves $y = x$ and $y = x^4$ is rotated about the line $x = 2$ and $y = -1$. What are the volumes?







$y_1 = x_1 \qquad y_2 = x_2^4$

about $x = 2$

$$V = \int_0^1 \pi\left[(2-x_1)^2 - (2-x_2)^2\right] dy$$

$$= \pi\int_0^1 \left((2-y)^2 - (2-y^{1/4})^2\right) dy$$

$$= \pi\int_0^1 \left[\cancel{4} - 4y + y^2 - \cancel{4} + 4y^{1/4} - y^{1/2}\right] dy$$

$$= \pi\left[-2y^2 + y^3/3 + \frac{16}{5}y^{5/4} - \frac{2}{3}y^{3/2}\right]_0^1 = \frac{13\pi}{15}$$

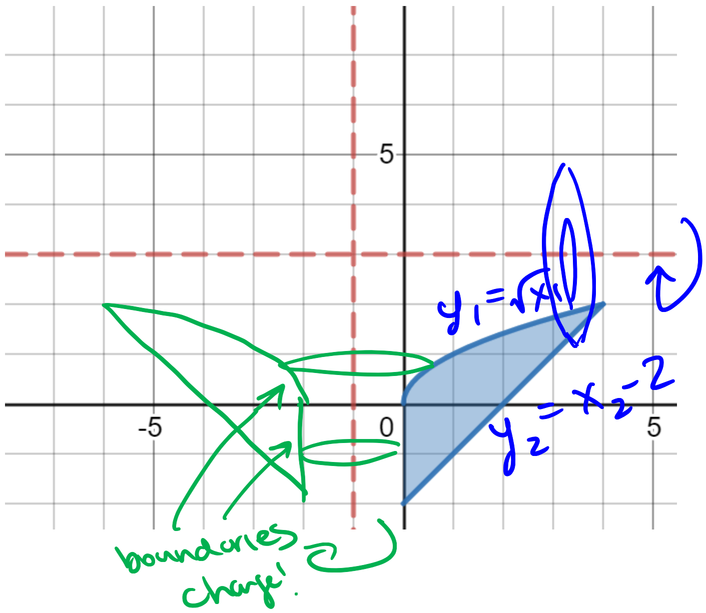
about $y = -1$

$$V = \pi\int_0^1 \left((y_1+1)^2 - (y_2+1)^2\right) dx = \pi\int_0^1 \left((x+1)^2 - (x^4+1)^2\right) dx$$

$$= \pi\int_0^1 (x^2 + 2x + \cancel{1} - x^8 - 2x^4 - \cancel{1})\, dx$$

$$= \pi\left[\frac{x^3}{3} + x^2 - \frac{x^9}{9} - \frac{2}{5}x^5\right]_0^1 = \frac{37\pi}{45}$$

Practice: Determine the volume of the shape made by rotating the region bound between $y = \sqrt{x}$ and $y = x - 2$ and the y-axis after rotating it around the line $x = -1$ and $y = 3$



about $y = 3$

$$V = \int_0^4 \pi\left[(3-y_2)^2 - (3-y_1)^2\right] dx$$

$$= \int_0^4 \pi\left[(3 - x + 2)^2 - (3 - \sqrt{x})^2\right] dx$$

$$= \pi\int_0^4 \left(25 - 10x + x^2 - 9 + 6\sqrt{x} - x\right) dx$$

$$= \pi\left[16x - \frac{11}{2}x^2 + \frac{x^3}{3} + 4x^{3/2}\right]_0^4$$

$$= \frac{88\pi}{3}$$

about $x = -1$

$$V = \int_0^2 \pi\left[(x_2+1)^2 - (x_1+1)^2\right] dy + \int_{-2}^0 \pi\left[(x_2+1)^2 - 1^2\right] dy$$

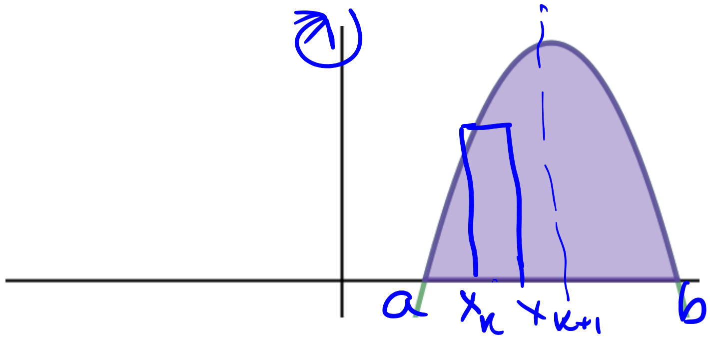
$$= \pi\int_0^2 \left((y+3)^2 - (y^2+1)^2\right) dy + \pi\int_{-2}^0 \left((y+3)^2 - 1\right) dy$$

$$= \pi\int_0^2 \left(y^2 + 6y + 9 - y^4 - 2y^2 - 1\right) dy + \pi\int_{-2}^0 \left(y^2 + 6y + 8\right) dy$$

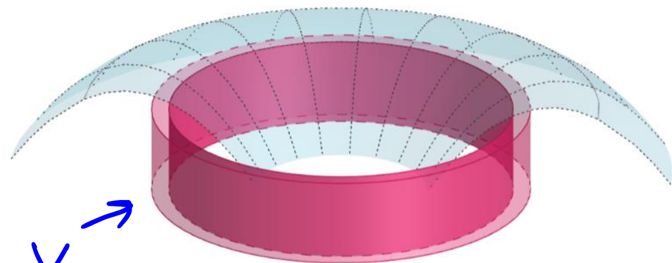
$$= \pi\left[-\frac{y^3}{3} + 3y^2 + 8y - \frac{y^5}{5}\right]_0^2 + \pi\left[\frac{y^3}{3} + 3y^2 + 8y\right]_{-2}^0$$

$$= \frac{128}{5}\pi$$

Volume using Shells:
Given a region bound between the function $y = f(x)$ (not 1-to-1) and the x-axis on the interval $x \in [a, b]$ and rotated around the y-axis, define a Riemann sum using cylinders whose limit is the desired volume.



V_k



$2\pi r h = SA$

$$V = \lim_{n\to\infty} \sum V_k \quad , V_k \text{ voluem of } k^{th} \text{ cylinder}$$

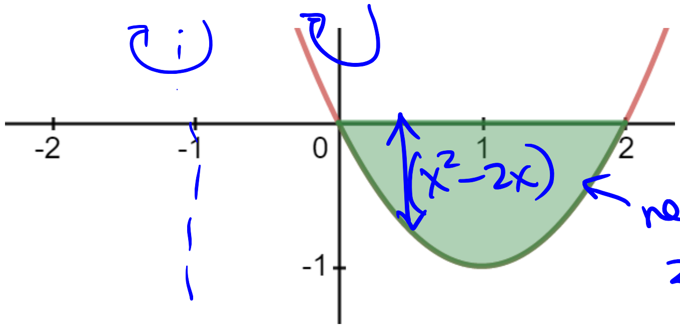
$$= \lim_{n\to\infty} \sum 2\pi f(c_k) x_k \Delta x$$

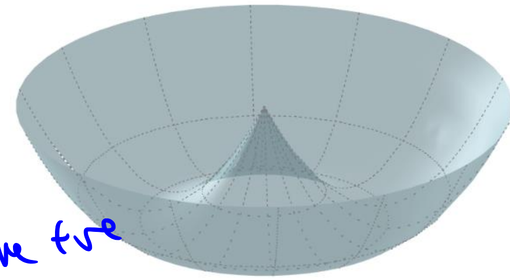
$$= \int_a^b \underbrace{2\pi x f(x)}_{\text{surface area of the cylinder}} dx$$

$$V_k = C_{out} - C_{in}$$
$$= \pi r_{out}^2 h - \pi r_{in}^2 h$$
$$= \pi x_{k+1}^2 h - \pi x_k^2 h$$
$$= \pi h \left[x_{k+1}^2 - x_k^2 \right]$$
$$= \pi h (x_{k+1} + x_k) \overbrace{(x_{k+1} - x_k)}^{\Delta x}$$
$$\downarrow$$
$$\pi h (x_k + x_k) \Delta x_k$$
$$= \pi f(c_k)(2x_k) \Delta x_k$$

$$c_k \in [x_k, x_{k+1}]$$

Practice: Determine the volume formed when the region bound between the curve $y = x^2 - 2x$ and the x-axis on the interval $x \in [0, 2]$ is rotated about the y-axis





negative so make +ve

$$V = \int_0^2 2\pi x f(x)\, dx = -\int_0^2 2\pi \left(x\,(x^2-2x)\right) dx$$

$$= -2\pi \int_0^2 (x^3 - 2x^2)\, dx$$

$$= -2\pi \left[\frac{x^4}{4} - \frac{2}{3}x^3\right]_0^2$$

$$= \frac{8\pi}{3}$$

Take the same area and rotate it about the line $x = -1$ instead. What is the new volume?

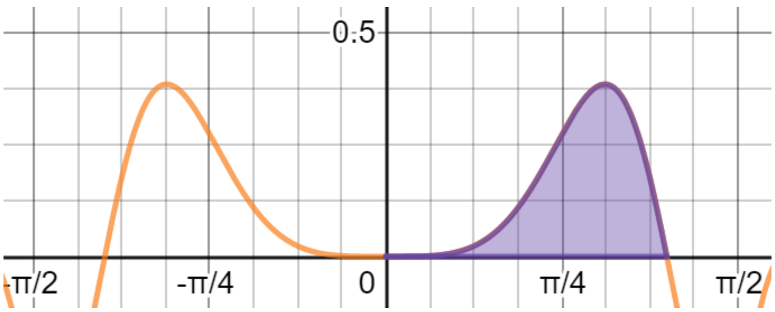
radius is now $x+1$

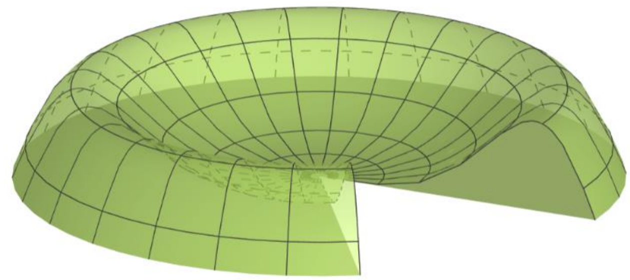
$$V = \int_0^2 2\pi (x+1)(2x - x^2)\, dx$$

$$= 2\pi \int_0^2 (-x^3 + x^2 + 2x)\, dx$$

$$= 2\pi \left[\frac{-x^4}{4} + \frac{x^3}{3} + x^2\right]_0^2 = \frac{16\pi}{3}$$

Practice: Determine the volume made when the region enclosed between $y = \sin^2 x^2 \cdot \cos x^2$ and the x-axis on the interval $x \in \left[0, \sqrt{\frac{\pi}{2}}\right]$ after being rotated about the y-axis.





$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \sin^2 x^2 \cdot \cos x^2 \, dx$$

$u = \sin x^2$

$du = \cos x^2 \cdot 2x \, dx$

$u(0) = 0 \quad u\left(\sqrt{\frac{\pi}{2}}\right) = 1$

$$= \int_0^1 \pi u^2 \, du$$

$$= \pi/3$$

Practice Problems: 7.3 # 21-26, 39-53	
Textbook Readings: 7.3 page 386-389	
Workbook Practice: page 338-349	
Next Class: Differential Equations	