Optimization

Goal: • Can interpret the zeros of the derivative of some function as the points the function reaches it maximum or minimum values. • Can create an equation for geometric objects and problems.
Terminology: • Optimization

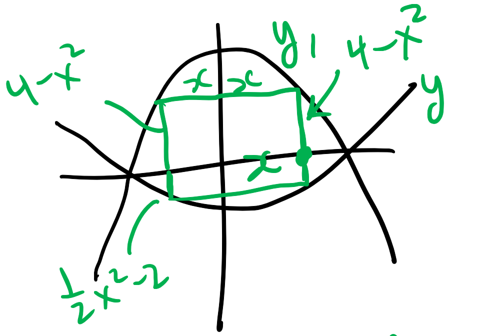
We know that local maximum and minimums occur for the function f when f' changes sign (First Derivative Test). Optimization is the application of finding max and minimums in order to maximize material used, or minimize cost to build.

General Strategy

- Draw a labeled picture! So important to give things good names and understand what you are actually looking at
- Identify your constraint and make an equation
- Make an equation for what you want to optimize in terms of one variable by using your constraint
- Find the derivative and find the extrema.

Example: What is the largest rectangle (in terms of area) that can fit between the parabolas?

$$y_1 = 4 - x^2$$
$$y_2 = \frac{1}{2}x^2 - 2$$



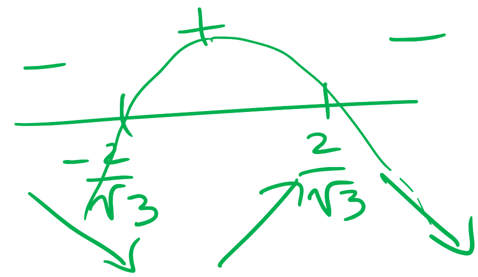
$A(1) = \left(3 + \frac{3}{2}\right)(2)$ ✓

max $A = h \times l$
$= (y_1 - y_2)(2x)$

$y_1 = 4 - x^2 \quad y_2 = \frac{1}{2}x^2 - 2$

$$A(x) = \left(4 - x^2 - \frac{1}{2}x^2 + 2\right)(2x)$$
$$= \left(6 - \frac{3}{2}x^2\right)(2x)$$
$$= 12x - 3x^3$$

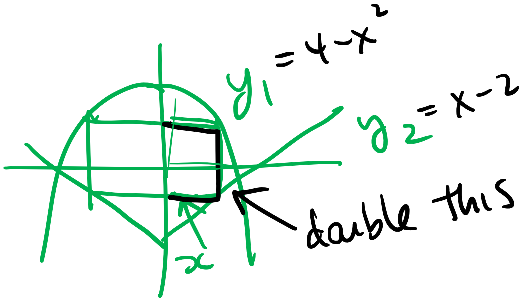
$$\Rightarrow \frac{d}{dx}A(x) = 12 - 9x^2 = 0$$
$$9x^2 = 12$$
$$x^2 = 4/3$$
$$x = \pm 2/\sqrt{3}$$



@ $x = \frac{2}{\sqrt{3}}$ we have a local max

Practice: What is the largest rectangle (in terms of perimeter) that can fit between the curves

$$y_1 = 4 - x^2$$
$$y_2 = |x| - 2$$



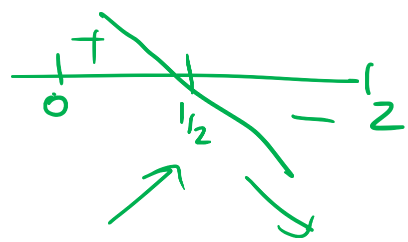
$P(0) = 12$

$P(2) = 8$

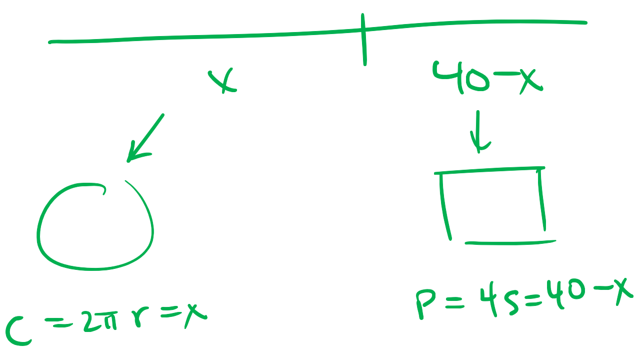
$P(\frac{1}{2}) = 12.5$

$$P(x) = 2\left(\cancel{x} + (2 - \cancel{x}) + 4 - x^2 + x\right) = 2(6 + x - x^2)$$
$$= 12 + 2x - 2x^2$$

$$\frac{d}{dx}P(x) = 2 - 4x = 0$$
$$4x = 2$$
$$\boxed{x = 1/2}$$



Practice: A piece of wire 40 cm long is cut and bent into a square and a circle. How should the wire be cut to (a) maximize total enclosed area (b) minimize total enclosed area.



$$A = \pi r^2 + s^2$$

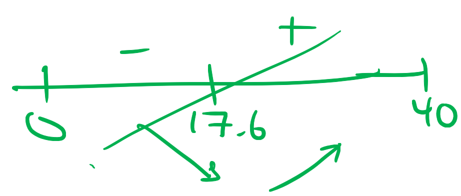
$$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(10 - \frac{x}{4}\right)^2$$

$$\frac{d}{dx}A(x) = \frac{\cancel{2\pi} x}{\cancel{2\pi}} \cdot \frac{1}{2\pi} + \cancel{2}\left(10 - \frac{x}{4}\right)\left(-\cancel{\frac{1}{4}}\right) -\frac{1}{2} = 0$$

$$\Rightarrow \frac{x}{2\pi} - 5 + \frac{x}{8} = 0$$

$$x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = 5$$

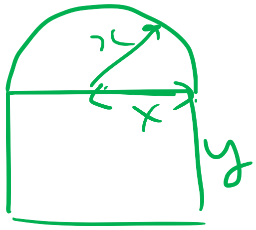
$$\boxed{\begin{array}{c} x = 17.6 \\ \text{minimizes} \end{array}}$$



$A(0) = 100$

$A(40) = 127 \longrightarrow$ make it all circle to maximize

Practice: A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



$P = \pi x + 2x + 2y$ const. $\quad y = \dfrac{P - \pi x - 2x}{2}$

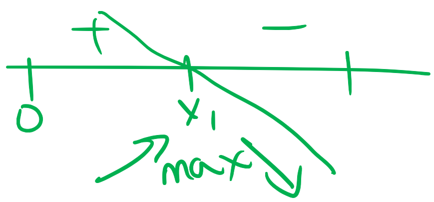
½ Area ◠ Area ▭

$L(x,y) = \frac{1}{4}(\pi x^2) + 2xy$

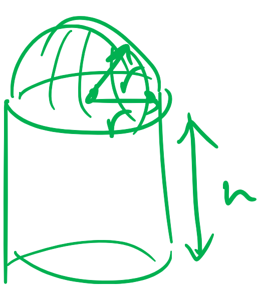
$\frac{d}{dx} L(x) = \frac{\pi x}{2} + 2y + 2x\left(\frac{-\pi - 2}{2}\right)$

$\Rightarrow 0 = \frac{\pi x}{2} + P - \pi x - 2x + (-\pi x - 2x)$

$0 = P - 4x - \frac{3\pi}{2}$ $\qquad \boxed{x_1 = \dfrac{P}{\frac{3\pi}{2} + 4}}$



Practice: A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.



$V = \frac{2}{3}\pi r^3 + \pi r^2 h$ const.

area of ▭ area ⊕

$C = 2\pi r h + (4\pi r^2)$ ← doubled

cost

$h = \dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2}$

$C = 2\pi r\left(\dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2}\right) + 4\pi r^2$

$C(r) = \frac{2V}{r} - \frac{4}{3}\pi r^2 + 4\pi r^2$

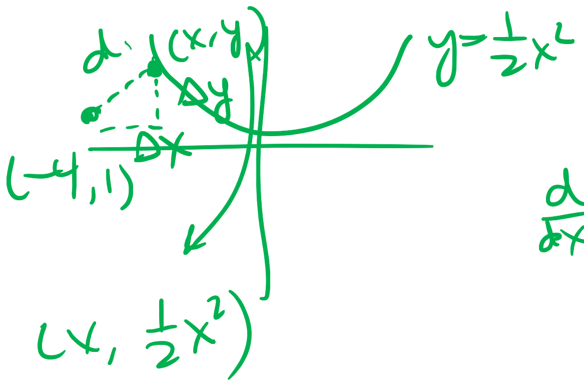
$\frac{d}{dr} C = \frac{-2V}{r^2} - \frac{8}{3}\pi r + 8\pi r = 0$

$-2V - \frac{8}{3}\pi r^3 + 8\pi r^3 = 0$

$r^3 = \dfrac{2V}{8\pi - 8\pi/3}$

$r_1 = \sqrt[3]{\dfrac{2V}{8\pi - \frac{8\pi}{3}}}$ min

Example: Find the point on the parabola $y = \frac{1}{2}x^2$ that is closest to the point $(-4, 1)$



$d^2 = \Delta x^2 + \Delta y^2$

$\frac{d}{dx}\left[d^2(x) = (x+4)^2 + (\frac{1}{2}x^2 - 1)^2\right]$

$2d(x)\, d' = 2(x+4) + 2(\frac{1}{2}x^2 - 1)(x)$

$= 0$ solve for x

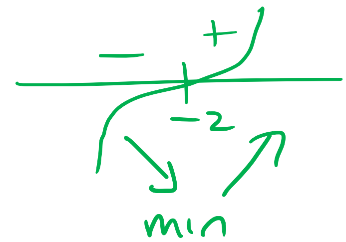
$\Delta x = (x+4)$

$\Delta y = (\frac{1}{2}x^2 - 1)$

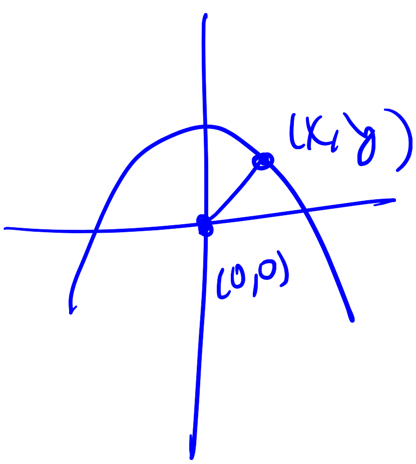
$\cancel{2x} + 8 + x^3 - \cancel{2x} = 0$

$x^3 = -8$

$\boxed{x = -2}$



Practice: What is the point on the parabola $y = 1 - x^2$ that is closest to the origin?



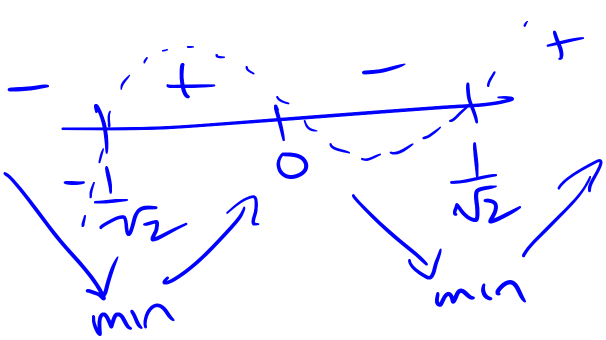
$d^2 = x^2 + (1 - x^2)^2$

$2d\, d' = 2x + 2(1 - x^2)(-2x)$

$0 = 2x - 4x + 4x^3$

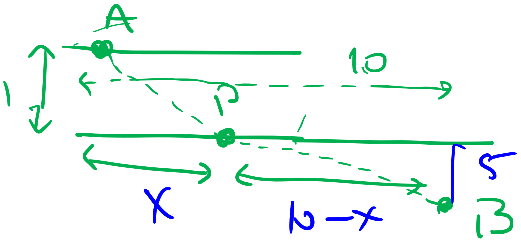
$0 = 2x(2x^2 - 1)$

$x = 0, \pm\frac{1}{\sqrt{2}}$



local min @ $x = \pm\frac{1}{\sqrt{2}}$

Practice: Fiber optics need to be laid between two communities. Community A is along a river that is 1 km wide, on the opposite side is community B which is 10 km downstream from A and 5 km inland. It costs \$300/m to install fibre optics under the river and \$200/m to install it on land. How far downstream from A should the cables be built?

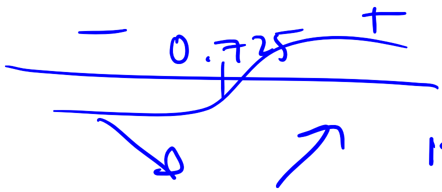


cost cost

$$C(x) = 3\sqrt{1+x^2} + 2\sqrt{25+(10-x)^2}$$

C is cost in hundreds

$$\frac{dC}{dx} = \frac{3x}{\sqrt{1+x^2}} - \frac{2(10-x)}{\sqrt{25+(10-x)^2}} = 0 \Rightarrow \frac{9x^2}{1+x^2} = \frac{4(10-x)^2}{25+(10-x)^2}$$

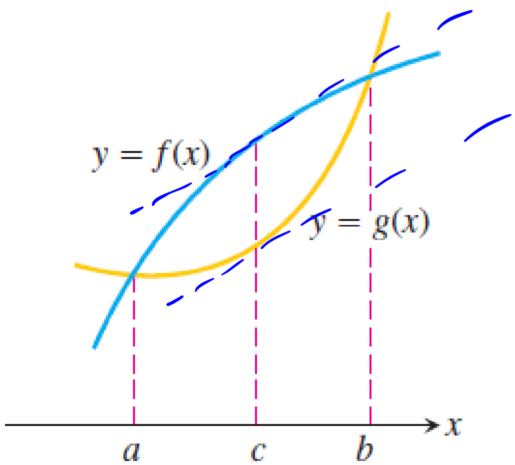


is a min

not fun, use a computer

$x = 0.725$ km

Practice: Let $f(x)$ and $g(x)$ be the differentiable functions graphed below. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at c? Give reasons for your answer.



$$V(x) = f(x) - g(x)$$

$$\frac{dV}{dx} = f'(x) - g'(x)$$

$$\left.\frac{dV}{dx}\right|_{x=c} = 0 = f'(c) - g'(c)$$

$$f'(c) = g'(c)$$

They have the same slope

Practice Problems: 4.4: # 5-17, 20-22, 26, 27, 31, 36, 42
Textbook Readings: 4.4 page 206-210
Workbook Practice: page 211-220
Next Day: Related Rates