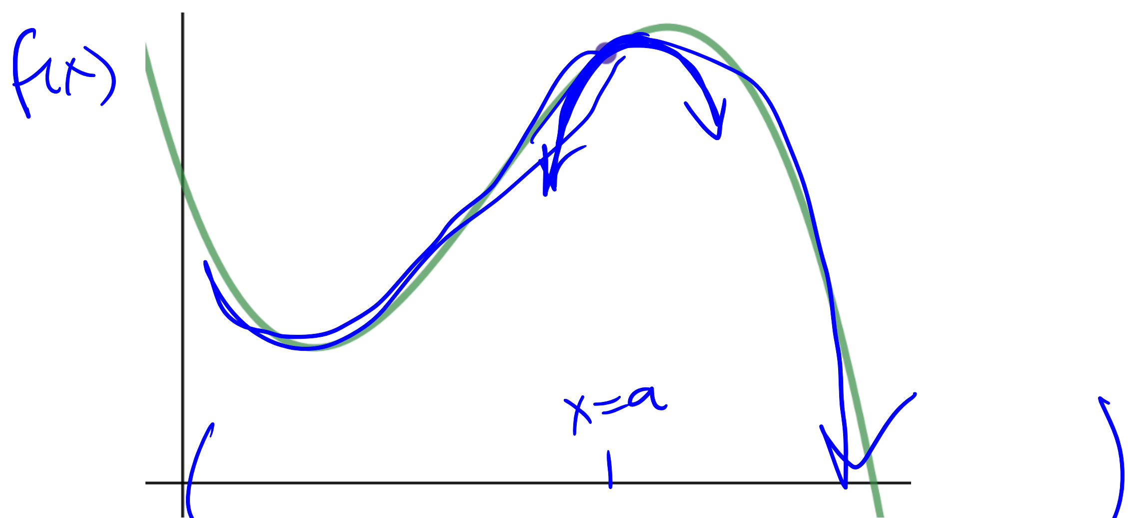
Linearization

Goal: • Understands linearization is just the tangent line at a point. • Understands that linearization “formula” is just point-slope form of tangent line.	
Terminology: • Linearization	

Our goal today is to approximate functions using polynomials and use our approximations to help analyze the curve.

Consider the general curve of $y = f(x)$ below



If we were to approximate $f(x)$ with a polyomial, $p(x)$, at the point $x = a$, what characteristics would you like p to have?

$p(a) = f(a)$

$p'(a) = f'(a)$ $\quad \dots \quad$ $p^{(n)}(a) = f^{(n)}(a)$

$p''(a) = f''(a)$

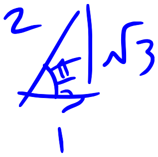
This is the process of **Linearization** and if we were to continue it would generate a **Taylor Series** of order n (degree of the polynomial)

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{y = y_0 + m(x - x_0) = L(x)} + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{3!} + \dots \quad \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Linearization (and Taylor series) are useful when we are okay with a little error around the point $x = a$.

Practice: Find the linearization of $\cos x$ around $x = \frac{\pi}{3}$ → find a line that looks like $\cos x$

$f(x) = \cos x$



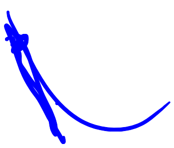
$f'(x) = -\sin x$

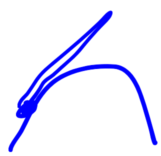
$f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ $\Rightarrow$ $y = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2} = L(x)$

Use the linearization to estimate $\cos\left(\frac{\pi}{3} + 0.02\right)$

$L\left(\frac{\pi}{3} + 0.02\right) = -\frac{\sqrt{3}}{2}(0.02) + \frac{1}{2} = 0.4826\ldots$

Is this an over-estimation or under-estimation?





$f''(x) = -\cos x$
$= -\frac{1}{2}$

concave up
underestimate

concave down
over estimate

$\Rightarrow$ overestimate.

Use a calculator to check the actual value of $\cos\left(\frac{\pi}{3} + 0.02\right)$

$0.48258\ldots$

Practice: Consider the relation

$$\frac{d}{dx}\left(x^2 - xy + y^3 = 1\right)$$

Linearize the relation around the point $(1, -1)$

$2x - y - xy' + 3y^2y' = 0$

$x = 1$
$y = -1$ $\Rightarrow$ $2 + 1 - y' + 3y' = 0$

$L(x) = -\frac{3}{2}(x - 1) - 1$

Use the linearization to estimate the solution to

$$1.1^2 - 1.1y + y^3 = 1$$

$$L(1.1) = \frac{-3}{2}(0.1) - 1 = -1.15$$

Are you over-estimating or under-estimating?

$$\frac{d}{dx}\left(2x - y - xy' + 3y^2 y'\right) = 0$$

$$2 - y' - y' - xy'' + 6y(y')^2 + 3y^2 y'' = 0$$

$$2 + 3 - y'' - \cancel{6}^{3}\left(\frac{9}{\cancel{4}2}\right) + 3y'' = 0$$

$$y'' > 0 \Rightarrow \text{under estimate}$$

Practice: If $f'(x) = \frac{\sqrt{x}}{x^3+1}$ and $f(1) = 2$, estimate $f(1.1)$. Is this an over-estimation or underestimation?

$$L(x) = \frac{1}{2}(x-1) + 2 \qquad L(1.1) = 2.05$$

$$f''(x) = \frac{\frac{1}{2\sqrt{x}}(x^3+1) - 3x^2\sqrt{x}}{(x^3+1)^2}$$

$$f''(1) = \frac{1-3}{4} < 0 \Rightarrow \text{overestimate}$$

Practice Problems: 4.5: # 1-14	
Textbook Readings: 4.5 page 220-222	
Workbook Practice: page 230-233, 238-239	
Next Day: Related Rates	

Related Rates (Preview)

Example: Consider a paper cone with height 10cm and radius 2cm that is full of water. A hole appears in the bottom and water begins to leave at a rate of 0.5cm^3/s. How fast is the height of water changing after 10 seconds?

Example: In New York on New Year's Eve, a giant ball of light will drop down a flag pole as people watch. The pole is 80 feet tall and you are 100 feet from the base of the flagpole. Assume your height is 6 feet. If the ball is dropping at rate of 1.3 feet/sec, how fast is the length of your shadow growing when the light is midway down the pole?