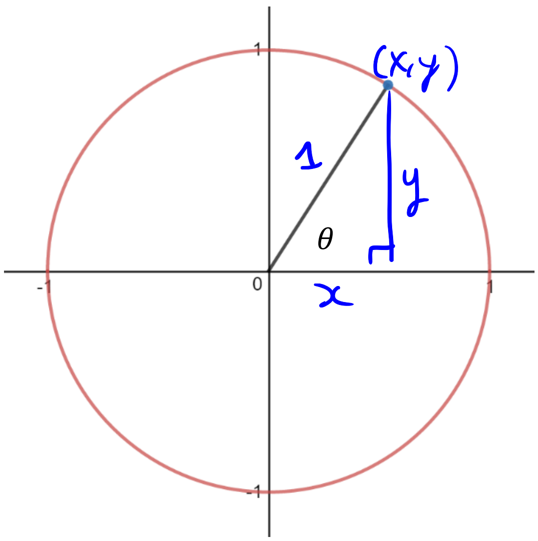
Trig Identities

Goal: • Can prove $\cos^2 x + \sin^2 x = 1$ using Pythagoras and can derive the other two identities from this one. • Understands that changing all trig terms to sine and cosine is the first step to simplifying identities. • Understands how the addition and subtraction identities are derived. Can use the identities to simplify trig statements.
New terminology: • Statement • Identity

Use the unit circle to show that $\boxed{\sin\theta = y \text{ and } \cos\theta = x}$.



$\cos\theta = \frac{x}{1} = x$

$\sin\theta = \frac{y}{1} = y$

$x^2 + y^2 = 1$

$\Rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1}$

Pythagorean Trig Identity

The above statement is called an **identity** – something that is fundamentally true and can be used to support other statements. In mathematics, a **statement** is something that is true or false.

$\cos^2\theta = 1 - \sin^2\theta$ $\cos\theta = \pm\sqrt{1-\sin^2\theta}$	$\sin^2\theta = 1 - \cos^2\theta$
$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$ $\boxed{\tan^2\theta + 1 = \sec^2\theta}$	$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$ $\boxed{1 + \cot^2\theta = \csc^2\theta}$

Test $x=1$ $\qquad \sin^2 1 = (\sin 1)^2$

Example: Prove or provide a counterexample to the following statements. If true, then show it.

$$\sin x = x - \frac{x^3}{6}$$

$\sin 1 = 0.84$

$1 - \frac{1}{6} = 0.83$

$\boxed{\text{False}}$

$$\sec x - \sec x \sin^2 x = \cos x$$

$\frac{1}{\cos 1} - \frac{1}{\cos 1} \cdot \sin^2 1 = 0.54\ldots$

$\cos 1 = 0.54$

$\sec x (1 - \sin^2 x) \overset{?}{=} \cos x$

$\frac{1}{\cos x}(1 - \sin^2 x) \overset{?}{=} \cos x$

$1 - \sin^2 x \overset{?}{=} \cos^2 x, \quad \cos x \neq 0$

Yep ✓

$\boxed{\text{True}}$

Practice: Prove or provide a counterexample to the following statement. If true, then show it.

$$\csc x - \sin x = \cot x$$

$$\tan^2 x \cdot \sin^2 x = \tan^2 x - \sin^2 x$$

$\tan^2 1 \cdot \sin^2 1 = 1.71\ldots$

$\tan^2 1 - \sin^2 1 = 1.71\ldots$

$\tan^2 x = \frac{\tan^2 x}{\sin^2 x} - 1, \quad \sin^2 x \neq 0$

$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} \div \sin^2 x - 1$

$= \frac{1}{\cos^2 x} - 1$

$\tan^2 x = \sec^2 x - 1$

Pythagorean Identity

$\boxed{\text{True}}$ for $\sin x \neq 0$

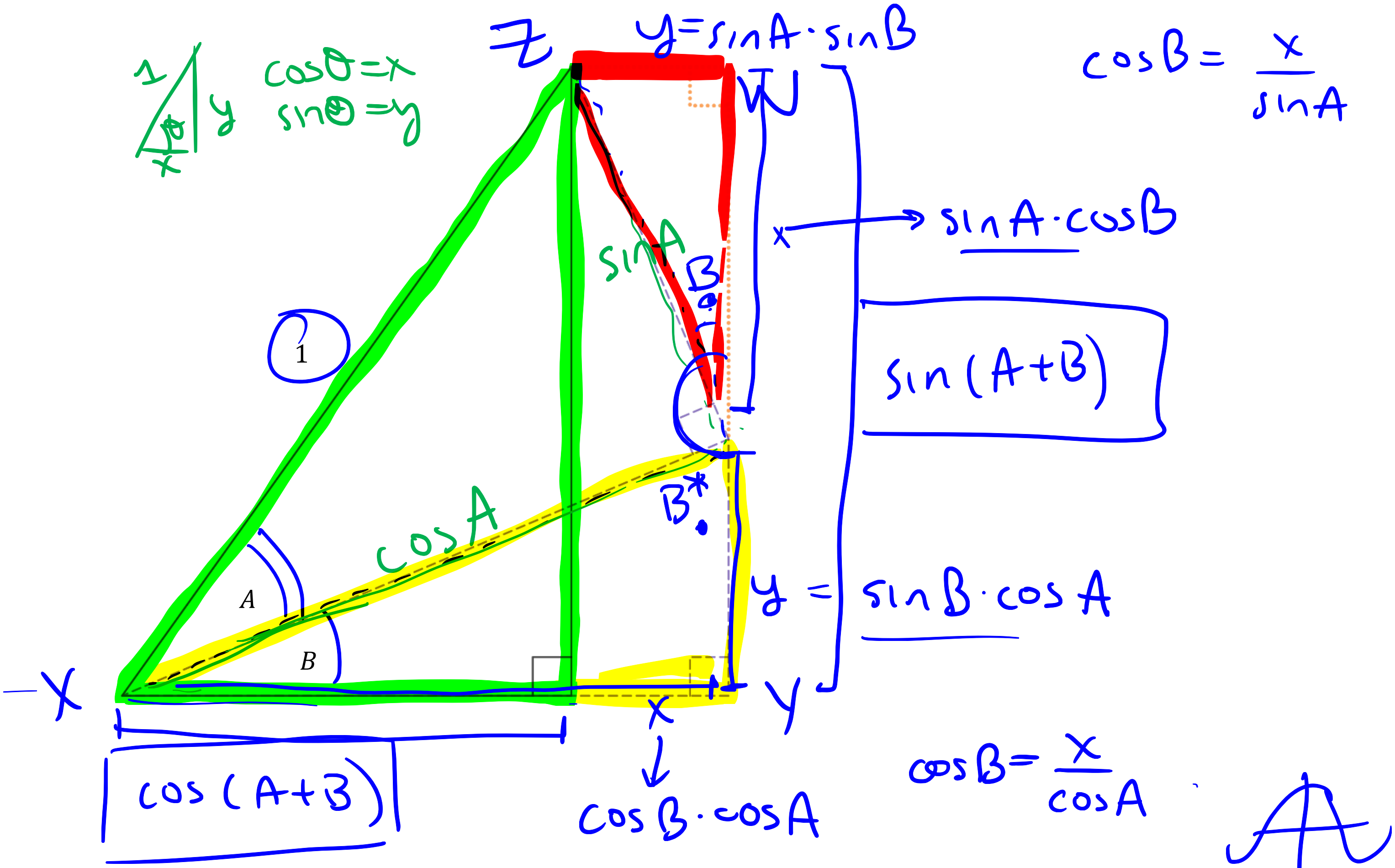
$\tan^2 x (1 - \cos^2 x)$

$= \tan^2 x - \tan^2 x \cdot \cos^2 x$

$= \tan^2 x - \frac{\sin^2 x}{\cancel{\cos^2 x}} \cdot \cancel{\cos^2 x}$

$= \tan^2 x - \sin^2 x$

Aside from the Pythagorean Identity, we will look at one other important identity: **Angle Addition Identity.**



We want to manipulate the above image so we know the following:

$\sin(A \pm B) =$ $\sin A \cos B \pm \sin B \cos A$

$\cos(A \pm B) =$ $\cos A \cos B \mp \sin A \sin B$

$\cos(-x) = \cos x$

$\sin(-x) = -\sin x$

Example:

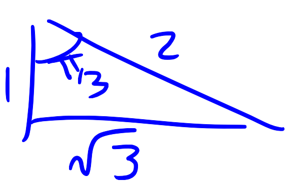
Determine the exact value of $\sin\left(\frac{\pi}{12}\right)$

$\boxed{\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}}$

$\frac{5\pi}{12}, \frac{7\pi}{12}$

$\frac{11\pi}{12}$

$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$



$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

$\frac{\pi}{2} + \frac{\pi}{3} - \frac{\pi}{4}$

Practice: Determine the exact value of $\cos\left(\frac{7\pi}{12}\right)$

$\frac{7\pi}{12} = \frac{7\pi}{3} - \frac{7\pi}{4}$; $\frac{\pi}{3} + \frac{\pi}{4}$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{7\pi}{3} - \frac{7\pi}{4}\right)$$

$$= \cos\left(\frac{7\pi}{3}\right)\cos\left(\overset{IV}{\frac{7\pi}{4}}\right) + \sin\left(\frac{7\pi}{3}\right)\sin\left(\overset{IV}{\frac{7\pi}{4}}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}}$$

Example: Simplify the following into a single sinusoidal equation

$\boxed{r\cos A}\sin x + \boxed{r\sin A}\cdot\cos x$

$\left(\sin(\cancel{A}+B) = \sin\cancel{A}\underbrace{\cos B}_{\#} + \cos\cancel{A}\underbrace{\sin B}_{\#}\right)$

$r\left(\cos A\sin x + \sin A\cos x\right)$

$r\cdot\sin(A+x) = r(\sin A\cos x + \cos A\sin x)$

Example: Determine an equation for $2\sin x - \cos x$

$r\sin(A+x) \leftarrow \boxed{r\cos A}\sin x \boxed{+ r\sin A}\cos x$

$= \boxed{2}\sin x \boxed{-1}\cos x$

$\boxed{}x^2\boxed{-4}x \qquad \boxed{a}x^2\boxed{+b}x\boxed{+c}$

$\Rightarrow r\cos A = 2 \qquad r\sin A = -1$

$r = \frac{2}{\cos A} \qquad r = \frac{-1}{\sin A} \qquad \Rightarrow \frac{2}{\cos A} = \frac{-1}{\sin A}$

$r = \frac{2}{\cos(-0.46)} = 2.236 \qquad \frac{\sin A}{\cos A} = -\frac{1}{2} = \tan A$

$A = \arctan\left(-\frac{1}{2}\right)$
$= -0.46$

$\Rightarrow 2.236\sin(x - 0.46)$

$r\cos A\sin x + r\sin A\cos x = r\sin(A+x)$

Practice: Determine an equation for $-3\sin x + 4\cos x$

$-3 = r\cos A$, $4 = r\sin A$

$r = \frac{-3}{\cos A} = \frac{4}{\sin A} \Rightarrow \tan A = -\frac{4}{3} \Rightarrow A = -0.927$

$r = -5$

$\Rightarrow \quad -5\sin(x - 0.927)$

$\cos^2 A + \sin^2 A = 1$

From the sum of angle identities, we get a set of important identities called **double angle identities**

$\sin 2A = \sin A\cos A + \sin A\cos A$

$= 2\sin A\cos A$

$\boxed{\cos 2A} = \cos A\cos A - \sin A\sin A$

$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$

$= 1 - \sin^2 A - \sin^2 A$

$\boxed{\cos 2A = 1 - 2\sin^2 A}$ $\qquad 1 - 2(1-\cos^2 x)$

$\boxed{\cos 2A = 2\cos^2 A - 1}$

$\boxed{\cos^2 A = \frac{1+\cos 2A}{2}}$ $\qquad$ $\boxed{\sin^2 A = \frac{1-\cos 2A}{2}}$

Calculus

Suggested Practice Problems:	6.1 # 3-6, 10-12, 14-16 6.2 # 1-8, 11, 14-16, 18-20, 23, 24
Textbook Reading:	page 290-295 and 299-305 Key Ideas page 296 and 305
Next Class:	Proving Trig Identities