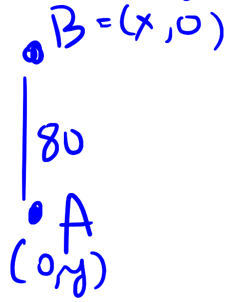
Related Rates

Goal: • Can create an equation to model problems based in geometry – Pythagoras, trig, surface area and volume. • Can differentiate an equation and relate how the rates depend on each other and make a new differential equation.
Terminology: • Related Rate

Distances

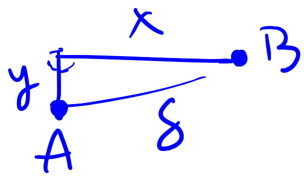
Example: At 1:00pm ship A was 80 km south of ship B. Ship A is sailing north at 30 km/h and ship B is sailing east 40 km/h. How fast is the distance between them changing at 3:00pm?

@ 1pm $\frac{dx}{dt} = 40$



$\frac{dy}{dt} = 30$

@ 3pm

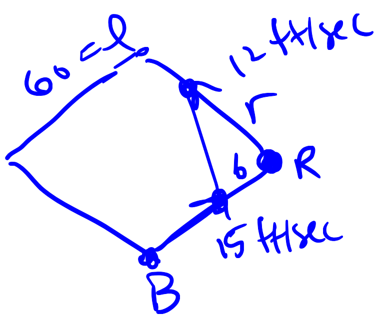


$$\frac{d}{dt}\left(\delta^2 = x^2 + y^2\right)$$

$$\cancel{2}\delta \frac{d\delta}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$\left.\frac{d\delta}{dt}\right|_{\substack{x=80\\ y=-20\\ \delta=82.5}} = \frac{80(40)+30(-20)}{82.5} = \underline{\underline{31.5\text{ km/h}}}$$

Example: A baseball diamond has 4 bases that make a diamond with edge lengths of 60 feet. There is a runner on first when the batter hits a fair ball. The batter runs to first at 15 feet/sec and the runner on first runs to second at 12 feet/sec. How fast is the distance between the batter and runner changing after 2 seconds when the runner is 30 feet from first?



$$\delta^2 = r^2 + b^2 \qquad \frac{db}{dt} = -15 \qquad \frac{dr}{dt} = 12$$

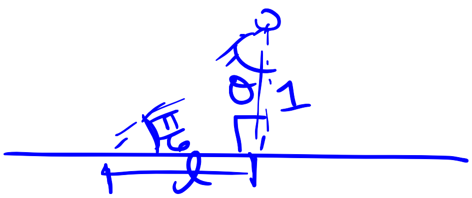
$$\cancel{2}\delta\frac{d\delta}{dt} = \cancel{2}r\frac{dr}{dt} + \cancel{2}b\frac{db}{dt}$$

$$\left.\frac{d\delta}{dt}\right|_{t=2} = \frac{24(12)+30(-15)}{38.4}$$

$$= \underline{\underline{-4.2\text{ ft/sec}}}$$

Angles

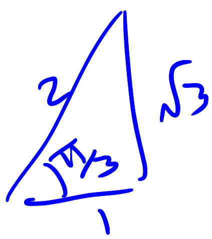
Example: A lighthouse is on an island 1 km from the shoreline. The light makes 1 revolution every 4 seconds. How fast is the lightbeam travelling across the shoreline when the angle the lightbeam makes with the shore is $\frac{\pi}{6}$ radians?



$\frac{d\theta}{dt} = \frac{2\pi}{4}$ rad/sec

$\tan\theta = l$

$\sec^2\theta \frac{d\theta}{dt} = \frac{dl}{dt}$

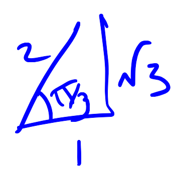


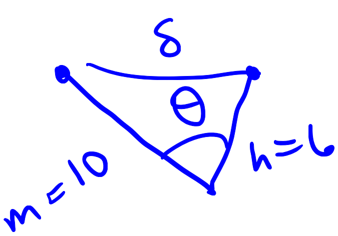
$\left.\frac{dl}{dt}\right|_{\theta=\frac{\pi}{3}} = \sec^2\left(\frac{\pi}{3}\right)\left(\frac{\pi}{2}\right)$

$= 4\left(\frac{\pi}{2}\right) = \underline{\underline{2\pi \text{ km/sec}}}$

Example: A clock's minute hand is 10cm and the hour hand is 6cm. If the angle between them is decreasing at a rate of 0.1 rad/minute how fast is the distance between the tip of each hand decreasing when the angle is $\frac{\pi}{3}$ radians?

$\frac{d\theta}{dt} = -0.1$ rad/min





$\delta^2 = m^2 + h^2 - 2mh\cos\theta$

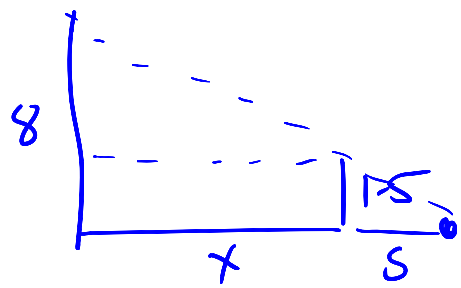
$\delta = 8.7$

$\cancel{2}\delta \frac{d\delta}{dt} = \cancel{2}mh\sin\theta \frac{d\theta}{dt}$

$\left.\frac{d\delta}{dt}\right|_{\theta=\frac{\pi}{3}} = \frac{60\sin\left(\frac{\pi}{3}\right)(0.1)}{8.7} = \underline{\underline{0.6 \text{ cm}}}/\text{min}$

Shadows:

Example: A person who is 1.5m tall walks toward a lightpost that is 8m tall at a constant speed of 2m/s. How fast is the length and tip of the persons shadow moving when the person is 4m from lightpost?



$\frac{dx}{dt} = -2$

$\frac{x}{6.5} = \frac{s}{1.5} = \frac{x+s}{8}$

$1.5x = 6.5s$

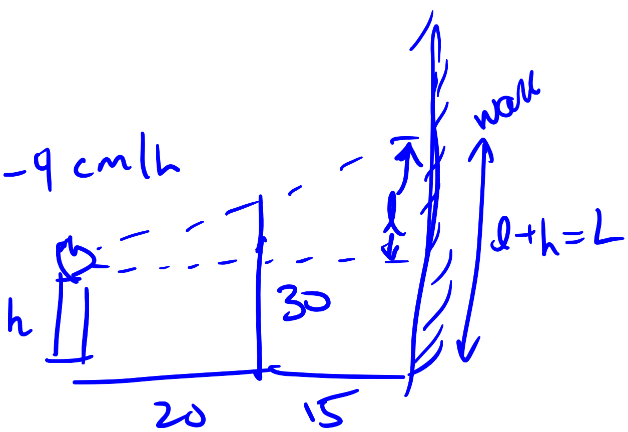
$1.5\frac{dx}{dt} = 6.5\frac{ds}{dt}$

$\frac{ds}{dt} = -0.46$ m/s (shadow length)

$\frac{ds}{dt} + \frac{dx}{dt} = -2.96$ m/s (shadow tip)

Example: A candle is placed a distance 20cm from a thin block of wood of height 30cm. The block is a distance 15cm from a wall. The candle burns down so that the height of the flame, h, decreases at the rate of 4 cm/hr. Find the rate at which the length of the shadow cast by the block on the wall increases.

$\frac{dh}{dt} = -4$ cm/h



$\frac{l}{35} = \frac{30-h}{20}$

$20l = 1050 - 35h$

$20\frac{dl}{dt} = -35\frac{dh}{dt}$

L is shadow length.

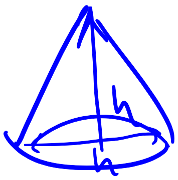
$\frac{dl}{dt} = 7$ cm/h

$\Rightarrow \boxed{\frac{dL}{dt} = 3 \text{ cm/h}}$

☆ Note these 2 don't depend on where the person or candle is.

Geometric Volumes:

Example: Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?

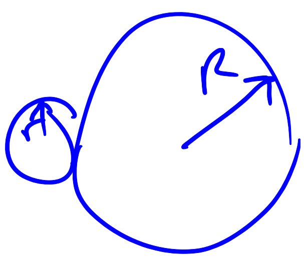


$$\frac{dV}{dt} = 15 \qquad V = \frac{1}{3}\pi r^2 h \qquad r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=3} = 2.12\text{ cm/sec}$$

Example: Two balloons are connected so the total volume is constant, but that air can travel between them. Determine how fast the radius of the smaller balloon is changing if the larger balloon's radius is shrinking 3 cm/sec at the moment the larger radius is two times the size of the smaller radius.



$$\frac{dR}{dt} = -3\text{ cm/sec} \qquad \left.\frac{dr}{dt}\right|_{R=2r} = ?$$

$$V = \frac{4\pi}{3}\left(r^3 + R^3\right) = \text{const.}$$

$$\frac{dV}{dt} = 0 = \frac{4\pi}{3}\left(3r^2\frac{dr}{dt} + 3R^2\frac{dR}{dt}\right)$$

$$\Rightarrow \frac{dr}{dt} = \frac{-R^2}{r^2}\frac{dR}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{R=2r} = 12\text{ cm/sec.}$$

Practice Problems: 4.6: # 8, 10-22, 24, 29-36
Textbook Readings: 4.6 page 232-236
Workbook Practice: page 221-229
Next Day: L'Hopital and Motion