FUNDAMENTAL THEOREM OF CALCULUS

(2009-6)
1.

The derivative of a function $f$ is defined by:

$$f'(x) = \begin{cases} g(x) & \text{for } -4 \leq x \leq 0 \\ 5e^{x/3} - 3 & \text{for } 0 < x \leq 4 \end{cases}$$

The graph of the continuous function $f'(x)$, shown in the figure above, has $x$-intercepts at $x = -2$ and $x = 3 \ln \frac{5}{3}$. The graph of $g$ on $[-4, 0]$ consists of a semi-circle, and $f(0) = 5$.

(a) For $-4 < x < 4$, find all values of $x$ at which the graph of $f$ has a point of inflection. Justify your answer.

(b) Find $f(-4)$ and $f(4)$.

(c) For $-4 \leq x \leq 4$, find the value of $x$ at which $f$ has an absolute maximum. Justify your answer.

(2010-5)
2.

The function $g$ is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of $g$, consists of a semi-circle and three line segments as shown on the figure above.

(a) Find $g(3)$ and $g(-2)$.

(b) Find the $x$-coordinate of each point of inflection of the graph of $y = g(x)$ on the interval $-7 < x < 5$. Explain your reasoning.

(c) The function $h$ is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the $x$-coordinate of each critical point of $h$, where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.
Let $f$ be a function that is continuous on the interval $[0, 4)$. The function $f$ is twice differential except at $x = 2$. The function $f$ and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of $f$ do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of $x$ at which $f$ has a relative extremum. Determine whether $f$ has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) Sketch the graph of a function that has all the characteristics of $f$.

(c) Let $g$ be the function defined by $g(x) = \int_{1}^{x} f(t) \, dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of $x$ at which $g$ has a relative extremum. Determine whether $g$ has relative maximum or a relative minimum at each of these values. Justify your answer.

(d) For the function defined in part (c), find all values of $x$, for $0 < x < 4$, at which the graph of $g$ has a point of inflection. Justify your answer.

The graph of the function $f$ shown above consists of a semicircle and three line segments. Let $g$ be the function given by $g(x) = \int_{-3}^{x} f(t) \, dt$

(a) Find $g(0)$ and $g'(0)$.

(b) Find all values of $x$ in the open interval $(-5, 4)$ at which $g$ attains a relative maximum. Justify your answer.

(c) Find the absolute minimum value of $g$ on the closed interval $[-5, 4]$. Justify your answer.

(d) Find all values of $x$ in the open interval $(-5, 4)$. At which the graph of $g$ has a point of inflection.
(2003-4)
5.

Let \( f \) be a function defined on the closed interval \([-3, 4]\) with \( f(0) = 3 \). The graph of \( f' \), the derivative of \( f \), consists of one line segment and a semicircle as shown.

(a) On what intervals, if any, is \( f \) increasing? Justify your answer.

(b) Find the \( x \)-coordinate of each point of inflection of the graph of \( f \) on the open interval \((-3, 4)\). Justify your answer.

(c) Find an equation for the line tangent to the graph of \( f \) at the point \((0, 3)\)

(d) Find \( f(-3) \) and \( f(4) \). Show the work that leads to your answers

(2011(B)-6)
6.

Let \( g \) be the piecewise linear function defined on \([-2\pi, 4\pi]\) whose graph is given above, and let
\[
f(x) = g(x) - \cos \frac{x}{2}
\]

(a) Find \( \int_{-2\pi}^{4\pi} f(x) \, dx \). Show the computations that lead to your answer.

(b) Find all \( x \)-values in the open interval \((-2\pi, 4\pi)\) for which \( f \) has a critical point.

(c) Let \( h(x) = \int_{0}^{3x} g(t) \, dt \). Find \( h' \left( -\frac{\pi}{3} \right) \).
7. The graph of the function $f$ shown above consists of six line segments. Let $g$ be the function given by $g(x) = \int_0^x f(t)dt$.

(a) Find $g(4), g'(4), g''(4)$.

(b) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.

(c) Suppose $f$ is defined for all real numbers and is periodic with a period of length 5. The graph above shows two periods of $f$. Given that $g(5) = 2$, find $g(10)$ and write an equation of the line tangent to the graph of $g$ at $x = 108$.

8. The graph of the function $f$ shown above consists of two line segments. Let $g$ be the function given by $g(x) = \int_0^x f(t)dt$.

(a) Find $g(1), g'(1), g''(1)$.

(b) For what values of $x$ in the open interval $(-2, 2)$ is $g$ increasing? Explain your reasoning.

(c) For what values of $x$ in the open interval $(-2, 2)$ is the graph of $g$ concave down? Explain your reasoning.

(d) Sketch the graph of $g$ on the closed interval $[-2, 2]$.
9. (a) Given \( 5x^3 + 40 = \int_c^x f(t) \, dt \)
   (i) Find \( f(x) \)

   (ii) Find the value of \( c \).

   (b) If \( F(x) = \int_x^3 \sqrt{1 + t^3} \, dt \), find \( F'(x) \).

10. Let \( f \) be a function whose domain is the closed interval \([0, 5]\).
    The graph of \( f \) is shown. Let \( h(x) = \int_0^{x^2 + 3} f(t) \, dt \)
    (a) Find the domain of \( h \).

    (b) Find \( h'(2) \).

    (c) At what \( x \) is \( h(x) \) a minimum? Show the analysis that leads to your conclusion.

11. | \( x \) | \(-2\) | \(-2 < x < -1\) | \(-1\) | \(-1 < x < 1\) | \(1\) | \(1 < x < 3\) | \(3\) |
    | \( f(x) \) | 12 | Positive | 8 | Positive | 2 | Positive | 7 |
    | \( f'(x) \) | -5 | Negative | 0 | Negative | 0 | Positive | \(\frac{1}{2}\) |
    | \( g(x) \) | -1 | Negative | 0 | Positive | 3 | Positive | 1 |
    | \( g'(x) \) | 2 | Positive | \(\frac{3}{2}\) | Positive | 0 | Negative | -2 |

The twice-differentiable functions \( f \) and \( g \) are defined for all real numbers \( x \). Values of \( f \), \( f' \), \( g \), and \( g' \) for various values of \( x \) are given in the table above.

(a) Find the \( x \)-coordinate of each relative minimum of \( f \) on the interval \([-2, 3]\). Justify your answers.

(b) Explain why there must be a value \( c \), for \(-1 < c < 1\), such that \( f''(c) = 0 \).

(c) The function \( h \) is defined by \( h(x) = \ln f(x) \). Find \( h'(3) \). Show the computations that lead to your answer.

(d) Evaluate
    \[
    \int_{-2}^{3} f'(g(x))g'(x) \, dx
    \]