Finding the Antiderivative

Goal: • Understand antiderivative is the backwards derivative • Have derivatives memorized so antiderivatives are also memorized • Can use simplifying techniques to find antiderivative: u-substitution, long division, completing the square
Terminology: • u-substitution

We know that

$$\int_a^x f(t)dt = F(x)$$

means that F is an antiderivative of f, that is if we differentiate F we get f:

$$\frac{d}{dx}F(x) = f(x)$$

The **indefinite integral is** not an area, but a symbol for the set of antiderivatives F

$$\int f(t)dt = F(x) + C \longrightarrow \text{any } C \in \mathbb{R}$$

$F(x)$	$\frac{d}{dx}F(x) = f(x)$	$\int f(t)dt = F(x) + C$
Polynomials		
$x^m, m \neq 0$	$\frac{d}{dx}x^m = mx^{m-1}$ $m \neq 0$	$\int mx^{m-1}dx = x^m + C$ let $m-1=n$, $n \neq -1$ $\Rightarrow \int x^n dx = (x^{n+1})/_{n+1} + C$
Exponential and Log		
e^x	$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x + C$
b^x	$\frac{d}{dx}b^x = \ln b \cdot b^x$	$\Rightarrow \int b^x dx = \frac{b^x}{\ln b} + C$

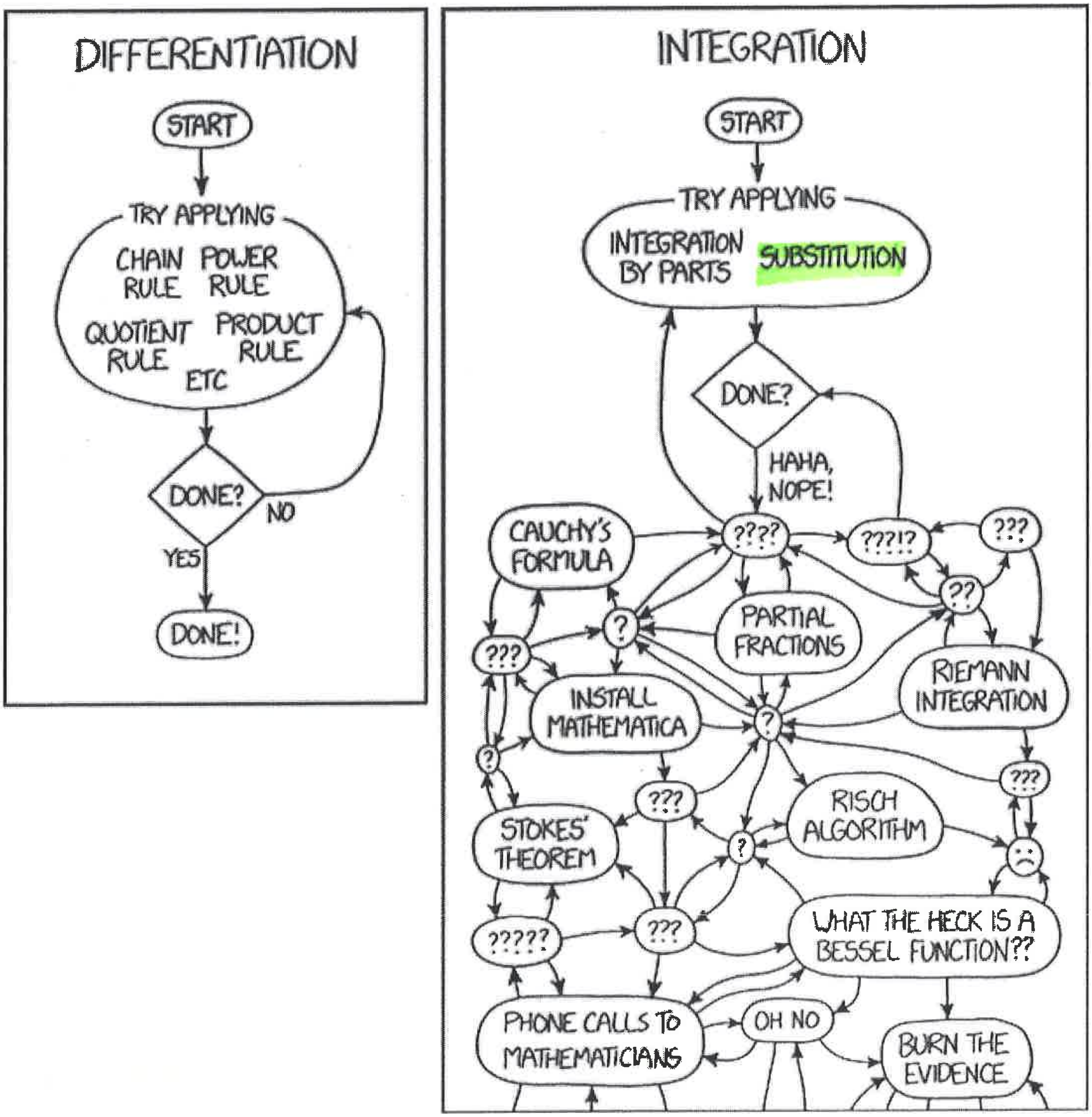
$F(x)$	$\frac{d}{dx}F(x) = f(x)$	$\int f(t)dt = F(x) + C$
$\ln x$	$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$ domain $x \neq 0$
$\log_b x$	$\frac{d}{dx}\log_b x = \frac{1}{\ln b \cdot x}$	same so use $\ln\lvert x\rvert$ $\int \frac{1}{x}dx = \ln b \cdot \log_b\lvert x\rvert + C$
Trig		
$\sin x$	$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\, dx = \sin x + C$
$\cos x$	$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\, dx = -\cos x + C$
$\tan x$	$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
$\sec x$	$\frac{d}{dx}\cancel{\tan}\sec x = \sec x\tan x$	$\int \sec x\tan x\, dx = \sec x + C$

$F(x)$	$\frac{d}{dx}F(x) = f(x)$	$\int f(t)dt = F(x) + C$
$\csc x$	$\frac{d}{dx}\csc x = \cancel{-\csc^2 x}$ $= -\csc x \cot x$	$\cancel{\int \csc^2 x\, dx} = -$ $\int \csc x \cot x\, dx = -\csc x + C$
$\cot x$	$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\, dx = -\cot x + C$
Inverse Trig		
$\arcsin x$	$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
$\arccos x$	$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$	same so use $\underline{\arcsin x}$ $\int \frac{dx}{\sqrt{1-x^2}} = -\arccos x + C$
$\arctan x$	$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \arctan x + C$
$\operatorname{arcsec} x$	$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}} = \operatorname{arcsec} x + C$

$F(x)$	$\frac{d}{dx}F(x) = f(x)$	$\int f(t)dt = F(x) + C$
$\text{arccsc}\, x$	$\frac{d}{dx}\text{arccsc}\, x = \frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$	use $\text{arcsec}\, x + C$
$\text{arccot}\, x$	$\frac{d}{dx}\text{arccot}\, x = \frac{-1}{1+x^2}$	use ~~[illegible]~~ $\arctan x + C$
	Derivative Rules	
$f \cdot g$	$f'g + g'f$	$\int (f'g + g'f)\, dx = fg + C$ NOT a product BC (parts)
$\frac{f}{g}$	$\frac{f'g - g'f}{g^2}$	$\int \frac{f'g-g'f}{g^2}\, dx = \frac{f}{g} + C$ NOT quotient rule
$f(g(x))$	$f'(g(x))g'(x)$	$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$ This <u>IS</u> a product

We really only know how to integrate (find an antiderivative) for these basic functions. If we are given any other function we need to transform it into something that looks like what is in the above table.

In general integration is complex. Almost an art because you need a certain amount of creativity to shape the integrand into something you can work with. There are a lot ~~of integration~~ of integration techniques of which we will learn a few but this is a comic by xkcd that does a good job of showing the difference between integration and differentiation



Source: https://xkcd.com/2117/

Technique 1: Substition (commonly called u substution)

The idea here is that $\int f(x)dx$ is hard, but $\int f(u(x))du$ is easier (because of chain rule!). We replace all of our x values with a u value instead.

Remember, where did the notation $f(x)dx$ come from?

$$\Delta F(x) = f(c)\,\Delta x \to \frac{\Delta F}{\Delta x} = f(c) \xrightarrow{\lim_{\Delta x \to 0}} \frac{dF}{dx} = f(x)$$

$$F(x) = \int f(t)\,dt \qquad dF = f(x)dx$$

Example: Use substitution to find du

$$x - 2 = u$$

$$\frac{d}{du}(x-2) = \frac{d}{du}(u)$$

$$\frac{dx}{du} = 1 \Rightarrow dx = du$$

Practice: Use substition to find du

$$\frac{d}{du}\left(\sin x = u\right)$$

$$\cos x \frac{dx}{du} = 1$$

$$du = \cos x\,dx$$

So we are going to use u-substitution when you see a function $u(x)$ and its derivative du in the integrand as a product

$$\int f(x)dx = \int f(u)du = \int \boxed{f(u(x))}\cdot u'(x)dx$$

compos. the derivative

Example:

$$\frac{1}{2}\int \frac{2(x+4)}{\sqrt{x^2+8x-9}}dx$$

good u

$$u = x^2 + 8x - 9$$

$$\frac{du}{du} = \frac{d}{du}(x^2+8x-9)$$

$$1 = (2x+8)\frac{dx}{du}$$

$$du = (2x+8)dx$$

$$= 2(x+4)dx$$

$$\Rightarrow \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}2u^{1/2} + C$$

$$= u^{1/2} + C$$

$$= \sqrt{x^2+8x-9} + C$$

Practice: Evaluate the integral

$$I = \int \frac{2x\ln(x^2+1)}{x^2+1}\,dx$$

$u = \ln(x^2+1)$

$du = \frac{2x}{x^2+1}\,dx$

$\int f(x)\,dx \to \int f(u)\,du$

$f(g(x))\,dx \to$

$u = g(x)$ $\quad \int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du}$

$du = g'(x)\,dx$

$\frac{d}{du} I = \frac{d}{du}\boxed{\frac{u^2}{2}} = u$

$$I = \int u\,du = \frac{u^2}{2} + C$$

$$= \frac{1}{2}\ln^2(x^2+1) + C$$

Technique 2: Substitution after writing in a different form

Example: Evaluate the integral

$$\int \tan x\,dx$$

$u = \tan x \quad du = \sec^2 x\,dx$

$u = 1 \quad du = 0$

$u = x \quad du = dx$

$$\int \tan x\,dx = \int \boxed{\frac{\sin x}{\cos x}\,dx}$$

$$= \int \frac{-du}{u}$$

$\to u = \sin x \quad \cancel{du = \cos x\,dx}$

$\to u = \cos x \quad \boxed{du = -\sin x\,dx}$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

Practice: Evaluate the integral

$$\int \sin^2 x \, dx$$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x - \sin^2 x = \cos 2x$

$2\sin^2 x = 1 - \cos 2x$

$\int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \int \frac{dx}{2} - \frac{1}{2}\int \cos 2x \, dx$ $\quad u = 2x$, $du = 2dx$

$= \frac{1}{2}x - \frac{1}{2}\int \cos u \frac{du}{2}$

$= \frac{1}{2}x - \frac{1}{4}\sin u + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

Technique 3: Substitution after long division.

Example:

$$\boxed{\int \frac{x^3 + 2x + 1}{x - 1} dx}$$

$\cancel{u = x-1 \quad du = dx}$

$\cancel{u = x^3 + 2x + 1 \quad du = (3x^2 + 2)dx}$

$$\begin{array}{r} x^2 + x + 3 \\ x-1 \overline{\smash{)}\, x^3 + 2x + 1} \\ -(x^3 - x^2) \\ \hline x^2 + 2x + 1 \\ -(x^2 - x) \\ \hline 3x + 1 \\ -(3x - 3) \\ \hline 4 \end{array}$$

$\Rightarrow \int \left(x^2 + x + 3 + \frac{4}{x-1}\right) dx$

$= \int (x^2 + x + 3) dx + 4\int \frac{dx}{x-1}$ $\quad u = x-1 \quad du = dx$

$= \frac{x^3}{3} + \frac{x^2}{2} + 3x + C_1 + 4\int \frac{du}{u}$

$= \frac{x^3}{3} + \frac{x^2}{2} + 3x + 4\ln|u| + C$

$= \frac{x^3}{3} + \frac{x^2}{2} + 3x + 4\ln|x-1| + C$

Practice: Evaluate the integral

$$\int \frac{x^4 - x^2 + x - 1}{x+4}\,dx$$

$$\begin{array}{r} x^3 - 4x^2 + 15x - 59 \\ x+4 \overline{)\, x^4 - x^2 + x - 1} \\ -(x^4 + 4x^3) \\ \hline -4x^3 - x^2 + x - 1 \\ -(-4x^3 - 16x^2) \\ \hline 15x^2 + x - 1 \\ -(15x^2 + 60x) \\ \hline -59x - 1 \\ -(-59x - 236) \\ \hline 235 \end{array}$$

$$\Rightarrow \int \left(x^3 - 4x^2 + 15x - 59 + \frac{235}{x+4}\right) dx$$

$$= \int (x^3 - 4x^2 + 15x - 59)\,dx + 235\int \frac{dx}{x+4}$$

$u = x+4$

$du = dx$

$$= \frac{x^4}{4} - \frac{4}{3}x^3 + \frac{15}{2}x^2 - 59x + 235\int \frac{du}{u} = \ln|u|$$

$$= \frac{x^4}{4} - \frac{4}{3}x^3 + \frac{15}{2}x^2 - 59x + 235\ln|x+4| + C$$

Technique 4: Substitution after completing the square (paired with trig)

Example:

$$\int \frac{dx}{x^2 - 6x + 13}$$

I want $(x-3)^2 = (2u)^2$

$x - 3 = 2u$

$dx = 2du$

$$\int \frac{dx}{(x-3)^2 + 4} = \int \frac{dx}{(x-3)^2 + 2^2}$$

$$= \int \frac{2\,du}{4u^2 + 4}$$

$$= \frac{2}{4}\int \frac{du}{u^2+1}$$

$$= \frac{1}{2}\arctan u + C$$

$$= \frac{1}{2}\arctan\left(\frac{x-3}{2}\right) + C$$

Practice: Evaluate the integral

$$\cancel{\int \frac{dx}{\sqrt{-x^2-6x}}} \quad \int \frac{dx}{\sqrt{-x^2-10x}}$$

$$\int \frac{dx}{\sqrt{-(x^2+10x+25)+25}} = \int \frac{dx}{\sqrt{25-(x+5)^2}}$$

want $(x+5)^2 = (5u)^2$

$x+5 = 5u$

$dx = 5du$

$$= \int \frac{5\,du}{\sqrt{25-25u^2}}$$

$$= \int \frac{\cancel{5}\,du}{\cancel{5}\sqrt{1-u^2}}$$

$$= \arcsin u + C$$

$$= \arcsin\left(\frac{x+5}{5}\right) + C$$

Practice Problems: 6.1 # 7-24 (what you need), 52
6.2 # 9-28 (what you need), 29, 30, 31-38 (what you need), 47, 48, 50, 51
6.1 # 60, 61