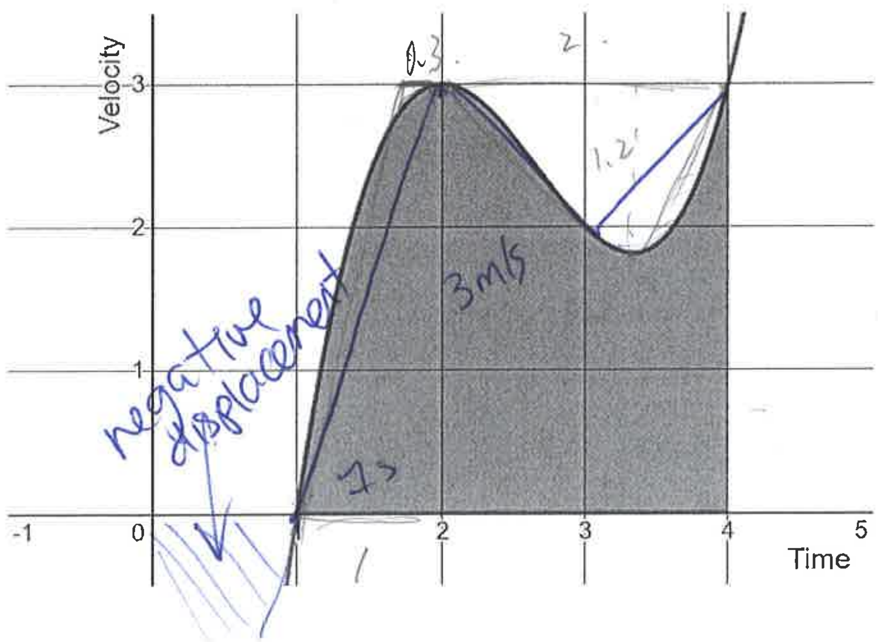
Approximating Area Under a Curve

Goal:
- Can approximate the area under a curve using geometry
- Understands how certain approximations may over or underestimate the actual area
- Can give a meaning to the area under a curve through application
- Use a for loop to create a recursive program on a graphing calculator

Terminology:
- Rectangular Approximation Method
- Trapezoid Rule/Method
- Reimann Sum
- Partition

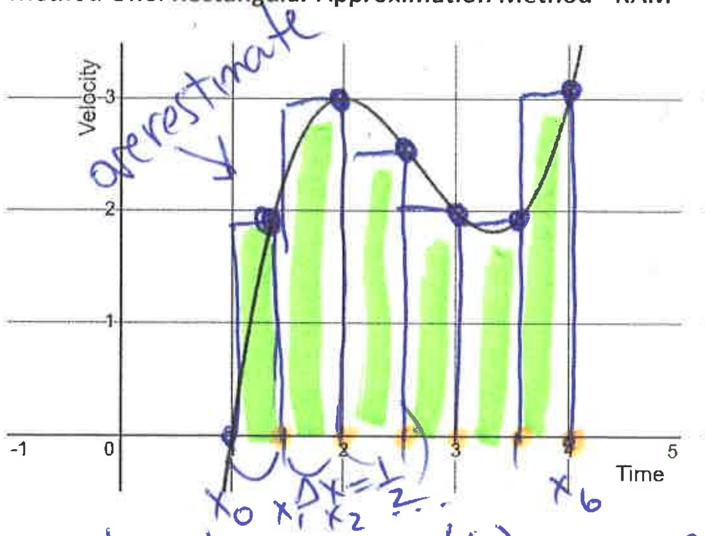
Reminder:
- Final Project second due date Monday January 13th. Choose the 1 problem that you will be presenting. You need to submit a rough draft of your presentation before Spring Break.
- Quiz Wednesday January 15th

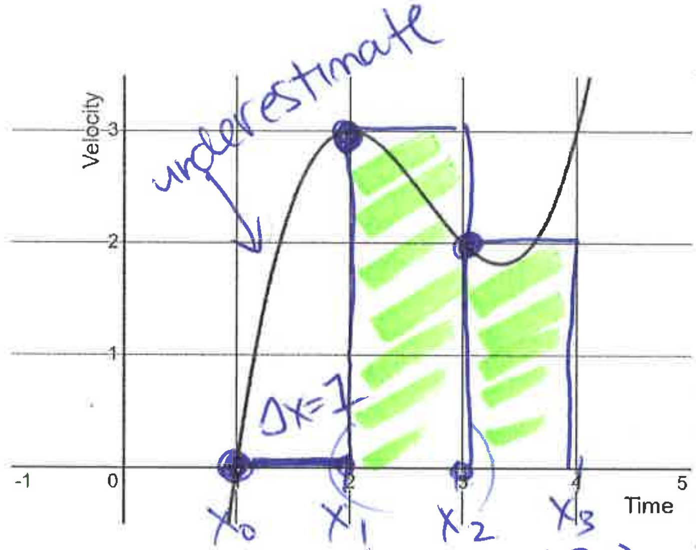
How can we estimate the shaded area under a curve within an error or 5% of the true value?



Group's Best Guess: → used triangles and rectangles to approximate!
circles too
6.75 m → displacement

Method One: Rectangular Approximation Method - RAM





→ cut into n (6) pieces
(the interval $[a,b] \Rightarrow [1,4]$)
and use the right endpoint to make the height.
⇒ RRAM

→ cut into n (3) pieces
and use left endpoint to make height
⇒ LRAM

width is the same, $\Delta x = \frac{b-a}{n}$

height depends on which side of the interval on the subinterval $[x_k, x_{k+1}]$

RRAM uses $f(x_{k+1})$ for the height

ie/ Area $= f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_6)\Delta x$

$\sim (1.9 + 3 + 2.5 + 2 + 1.9 + 3)\frac{1}{2} = 7.15$

LRAM uses $f(x_k)$ for the height.

ie/ Area $= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x$

$= (0 + 3 + 2)1 = 5$

Also ~~AREA~~ MRAM use $f\left(\frac{x_{k+1} + x_k}{2}\right)$ (the midpoint) for the height

Making a program for RAM

We are going to make a program in our calculators that will compute the RAM for the left, middle, and right sides so we understand how it works and so we don't need to do the same tedious calculations and can get a good approximation quickly. To start we will be using the following buttons a lot, make sure you can find them:

- ALPHA and A-LOCK
- PRGM and DRAW
- VARS
- TEST
- STO⇒

In PRGM move to NEW and Create New, then enter the name RAM. You will now be able to write the code :)

: ClrDraw , clears the graphing window

: FnOff , turn every function off in grapher

: FnOn 1 , turns on y_1 in grapher (see it)

: Prompt A,B,N , Interval [a,b] how many pieces n

: Disp "0 FOR LEFT", "1 FOR RIGHT", "0.5 FOR MIDDLE"

: Input T type of RAM

: (B-A)/N→D d is Δx

: 0→S → S is sum of area (starts @ 0)

PRGM : For(K,1,N,1) For loop begins a set of operations until we reach "end"
K changes, start at 1, end on N, add 1 to K

: A+(K-1)D→U U is the right endpoint on $[x_k, x_{k+1}]$

: U+TD→X X is left, middle or right of $[x_k, x_{k+1}]$

: Y_1→W $y_1(x) = W$ (height of rectangle)

DRAW : Line(U,0,U,W) Line connects coordinates $(x_k, 0)$ to $(x_k, y_1(x_k))$

: Line(U,W,U+D,W)

: Line(U+D,W,U+D,0)

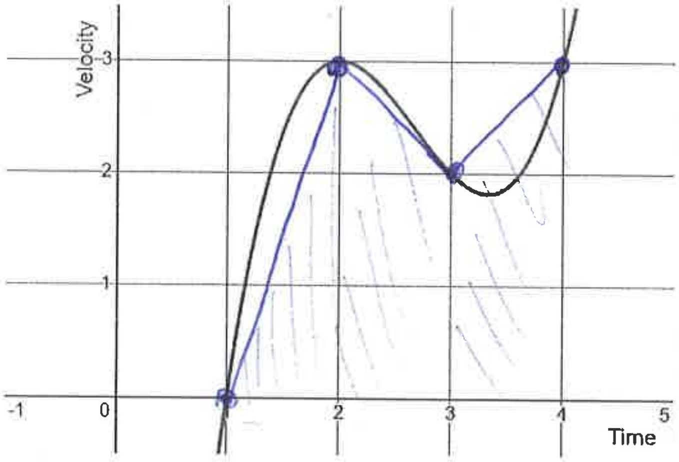
: S+DW→S $S_{old} + DW = S_{new}$

PRGM : End End the loop

PRGM : Pause Pause after graphing

: Disp "AREA IS", S

Method Two: Trapezoid Rule



cut into n pieces

width $\Delta x = \frac{b-a}{n}$

intervals

$[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$

for the subinterval $[x_k, x_{n+1}]$

the area of the trapezoid is avg height × base

$$\Rightarrow \text{Area} = \frac{f(x_{k+1}) + f(x_k)}{2} \Delta x$$

$$\Rightarrow \text{Total Area} = \left[\frac{f(x_1)+f(x_0)}{2} + \frac{f(x_2)+f(x_1)}{2} + \overset{\text{pair}}{\frac{f(x_3)+f(x_2)}{2}} + \ldots + \overset{\text{pair}}{\frac{f(x_n)+f(x_{n-1})}{2}} \right] \Delta x$$

$$= \left[\tfrac{1}{2} f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \tfrac{1}{2} f(x_n) \right] \Delta x$$

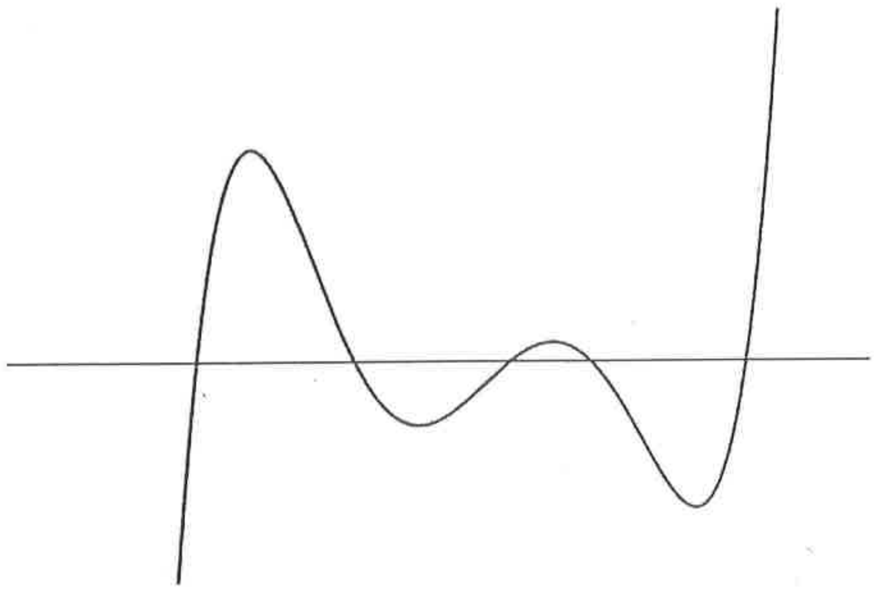
ie/ $\text{Area} = \left[\tfrac{1}{2} f(1) + f(2) + f(3) + \tfrac{1}{2} f(4) \right] \Delta x$

$$= \left[\tfrac{1}{2} 0 + 3 + 2 + \tfrac{1}{2} 3 \right] \cdot 1 = \cancel{8}\ 6.5$$

Weekend assignment is to make a program called "TRAP" that will compute the trapezoid approximation of the area.

What we have been making are specific instances of Riemann Sums, which is a general approach to find the area under a curve using rectangles. In the practice problems you will see that Trapezoid approximation is just an average of LRAM and RRAM.

We will be talking about the limit of Riemann Sums next class and I wanted to leave you with the general idea of a Riemann Sum. Read the text 5.2 page 258-260 for more detail.



Consider the function f and we want to find the net area under the curve on $[a, b]$. In general what we can do is consider some **partition** of $[a, b]$. That is, divide the interval intosome sequence $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ where $x_0 = a$ and $x_n = b$ and $x_k < x_{k+1}$.

With that our job is just to estimate the area under the curve on the subinterval $[x_k, x_{k+1}]$. If the partition is small enough (that is the largest subinterval has small length) then any regtangle in the subintervals will be a good approximation to the area.

So pick some arbitrary point in the subinterval, $c_k \in [x_k, x_{k+1}]$ and use $f(c_k)$ to make the height of the rectangle which has width $\Delta x_k = x_{k+1} - x_k$, hence it has area of

The net area will be

This is a general Riemann Sum. Observe that RAM uses $\Delta x = \frac{b-a}{n}$ and $x_{k+1} = x_k + \Delta x$ for each k. The differences in LRAM, MRAM, and RRAM come from the choice of c_k.

Practice Problems: 5.1 # 1-4, 5-8 (use $n = 2$ or 3 to do by hand, $n = 100$ using calculator), 9-13, 22, 24, 25

27, 28, 30

5.5 # 9, below questions (which are #1-6 but fixing notation)

23

Look Ahead: 5.2 the definite integral notation, how can we use limits to define the area under a curve?

(a) Use Trapezoid rule with $n = 4$ to approximate the area by hand.
(b) Predict whether this will be an overestimate or underestimate.
(c) Use Trapezoid rule with $n = 100$ to find the area using a program.

1. $f(x) = x$ on $[0, 2]$
2. $f(x) = x^2$ on $[0, 2]$
3. $f(x) = x^3$ on $[0, 2]$
4. $f(x) = 1/x$ on $[1, 2]$
5. $f(x) = \sqrt{x}$ on $[0, 4]$
6. $f(x) = \sin x$ on $[0, \pi]$