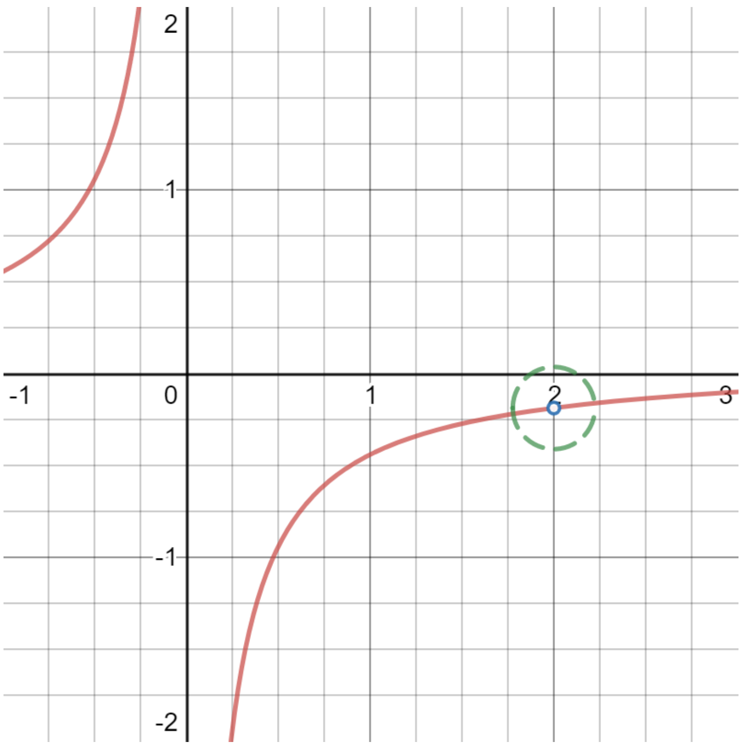
The Limit

We looked at finding the limit as the value the neighbourhood of a function approaches for arbitrarily small neighbuorhoods around a point $x = c$. Consider

$$\lim_{x\to 2} \frac{\frac{1}{x} - \frac{\sqrt{6-x}}{4}}{x-2}$$



Then the graph reveals that as we tighten up our neighbourhood around $x = 2$ that $f(x)$ looks like it approaches a value around -0.2.

A table of values gives us a better picture and we see that the limit is around -0.1875 (at least to four decimal places).

The only way to do better is to use algebra and simplify

$$\begin{aligned}\lim_{x\to 2} \frac{\frac{1}{x} - \frac{\sqrt{6-x}}{4}}{x-2} &= \lim_{x\to 2} \frac{4 - x\sqrt{6-x}}{4x(x-2)} \\ &= \lim_{x\to 2} \frac{16 - x^2(6-x)}{4x(x-2)\left(4 + x\sqrt{6-x}\right)} \\ &= \lim_{x\to 2} \frac{(x-2)(x^2-4x-8)}{4x(x-2)\left(4 + x\sqrt{6-x}\right)} \\ &= \lim_{x\to 2} \frac{x^2-4x-8}{4x\left(4 + x\sqrt{6-x}\right)} \\ &= -\frac{3}{16}\end{aligned}$$

Which happens to be -0.1875 exact!

x	1.99	1.999	1.9999	2.0001	2.001	2.01
$f(x)$	-0.18880	-0.18763	-0.18751	-0.18749	-0.18737	-0.18622

We say the limit does not exist if it does not approach a real number from both sides (left and right) and in the case above where there is an asymptote at $x = 0$ we could say that $\lim_{x\to 0^-} \frac{\frac{1}{x} - \frac{\sqrt{6-x}}{4}}{x-2} = \infty$ and $\lim_{x\to 0^+} \frac{\frac{1}{x} - \frac{\sqrt{6-x}}{4}}{x-2} = -\infty$ BUT keep in mind that infinity in not a number and we can not encircle it like we can real limit points.

Functions grow in the following order from fastest to slowest

1. Exponential: $y = e^x$
2. Polynomial: $y = x^n, n > 1$
3. Linear: $y = x$
4. Polynomial (radical): $y = x^n, 0 < n < 1$
5. Logarithmic: $y = \ln x$
6. Bounded (trig, constants, and exponential in the other direction – that is no growth): $y = e^{-x}$

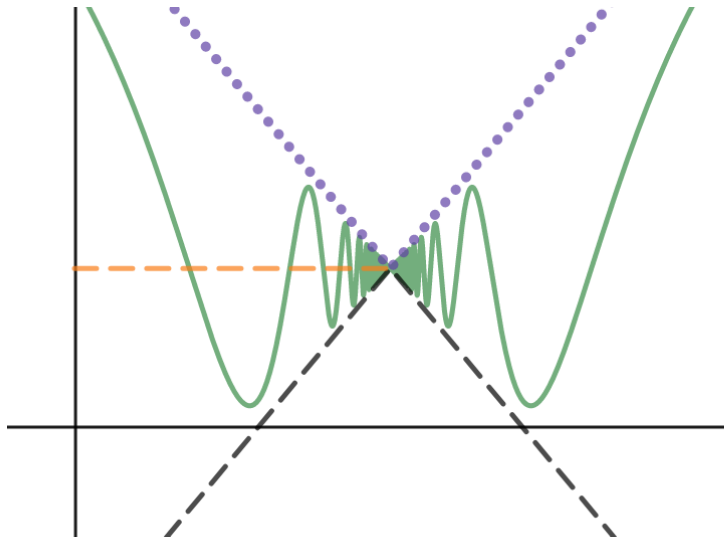
When adding or subtracting functions only the largest growth matters. When dividing we have three cases

a. $\frac{Fast}{Slow} \to \infty$

b. $\frac{Slow}{Fast} \to 0$

c. $\frac{same}{same} \to L = \frac{a}{b}$
Where a, b are related to the leading coefficients

We also used Sandwich Theorem to find limits of complex functions using simpler ones. If $g(x) < f(x) < h(x)$ for all $x \in (a, b)$ expect at some point $x = c$ AND $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} f(x) = L$



Namely we saw that $\lim_{A\to 0} \frac{\sin A}{A} = 1$ and $\lim_{A\to 0} \frac{1-\cos A}{A} = 0$

Things we need to know and understand:

- How to calculate the limit from a graph or table of values (with or without graphing calculator)
- How to calculate the limit using algebra and known limit values
- How to prove a limit using epsilon-delta
- That the limit exists when its one-sided limits approach the same value.
- Sandwich theorem is useful to find the limit of a complex function that is stuck between two simple functions.
- Limits at infinity depend on end behaviour that is determined by the order the function grows.

Review Questions:

1. What is a limit?
2. Why does $\lim_{x\to 0} \frac{|x|}{x}$ not exist?
3. Determine $\lim_{x\to 1} \frac{\frac{3}{2-x} - \sqrt{x+8}}{x-1}$
4. Determine $\lim_{x\to 0} 3x \cdot \csc(x^2 - 2x)$
5. Prove $\lim_{x\to 2} \frac{2}{x} = 1$
6. Use Sandwich Theorem to determine the values of $n > 0$ and m so that $\lim_{x\to\infty} \frac{2^{nx}+x^m}{8^{x-1}} = \begin{cases} 0 \\ 1 \\ \infty \end{cases}$
7. Use Sandwich Theorem to determine the values of $n > 0$ and m so that $\lim_{x\to-\infty} \frac{2^{nx}+x^m}{8^{x-1}} = \begin{cases} 0 \\ 1 \\ \infty \end{cases}$

Solutions:

1. Weak: A value (real number), L, that we can make arbitrarily close to the value of the function if we pick values of x close enough to the limit point.
   Strong. The limit of $f(x)$ as x approaches c is L if and only if $\forall\ \varepsilon > 0, \exists\ \delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - L| < \varepsilon$.
2. Because $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$ and $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$ which are not the same.
3. $\frac{17}{6}$

4. -1.5
5. We have that $|x-2|<\delta$ and we want $\left|\frac{2}{x}-1\right|<\varepsilon$ for any $\varepsilon>0$. Start with the function and its limit: $\left|\frac{2}{x}-1\right|=\frac{|2-x|}{x}=\frac{|x-2|}{x}<\frac{\delta}{x}$ as long as $\delta<2$ (since this guarantees that $x>0$) We now need to bound $\frac{1}{x}$ which effectively minimizes x. Since we currently have that $x\in(0,4)$ there is no good lower bound so we can make $\delta<1$ so that $x\in(1,3)$. Now $\frac{1}{x}<1$ and we have that $\left|\frac{2}{x}-1\right|<\frac{\delta}{x}<\delta$ so let $\delta=\min(\varepsilon,1)$ ■
6. First, m can be any real number, it won't matter since it is next to an exponential and the numerator grows $\mathcal{O}((2^n)^x)$. For the limit to be 0, we need $2^n<8$, hence $n<3$.
$$\lim_{x\to\infty}\frac{2^{nx}+x^m}{8^{x-1}}=8\cdot\lim_{x\to\infty}\frac{2^{nx}+x^m}{8^x}=8\cdot\lim_{x\to\infty}\frac{2^{nx}}{8^x}+8\cdot\lim_{x\to\infty}\frac{x^m}{8^x}=8\cdot\lim_{x\to\infty}\left(\frac{2^n}{8}\right)^x+8\cdot\lim_{x\to\infty}\frac{x^m}{8^x}$$
And the last piece will go to 0 via Sandwich Theorem.
$$0<8\cdot\lim_{x\to\infty}\frac{x^m}{8^x}<8\cdot\lim_{x\to\infty}\frac{2^x}{8^x}=0$$
Likewise, for it to blow up we need $2^n>8$, so $n>3$.
$$\lim_{x\to\infty}\frac{2^{nx}+x^m}{8^{x-1}}=8\cdot\lim_{x\to\infty}\frac{2^{nx}+x^m}{8^x}>8\cdot\lim_{x\to\infty}\frac{2^{nx}}{8^x}=8\cdot\lim_{x\to\infty}\left(\frac{2^n}{8}\right)^x=\infty$$
We can never make the limit 1 as if $n=3$ the limit is 8.
7. Same as above but now the denominator is going to 0 and the top grows $\mathcal{O}(x^m)$ so n can be any positive number. As long as $m\geq 0$ the numerator will grow larger (or at least be 1) while the denominator will approach 0^+ and the overall limit will go to infinity.
$$\lim_{x\to-\infty}\frac{2^{nx}+x^m}{8^{x-1}}>\lim_{x\to-\infty}\frac{x^m}{8^{x-1}}\geq\lim_{x\to-\infty}\frac{1}{8^{x-1}}=\infty$$
If $m<0$ then we can change the limit to
$$\lim_{x\to-\infty}\frac{2^{nx}+x^m}{8^{x-1}}>\lim_{x\to-\infty}\frac{x^m}{8^{x-1}}=\lim_{x\to\infty}\frac{8^{x+1}}{x^{|m|}}>\lim_{x\to\infty}\frac{8^{x+1}}{2^x}=\infty$$

Continuity

We used the definition of continuity to be the following:

f is continuous at $x=c$ if and only if $\lim_{x\to c}f(x)=f(c)$

This also requires that $f(c)$ is well defined and that $\lim_{x\to c}f(x)$ exists, and this gives us a way to describe familiar discontinuities. If there is an asymptote or a jump, then the limit does not exist, and if there is a hole (removeable discontinuity) then the function is not defined or does not equal the limit.

Once we have continuity, we have a fantastic result called **Intermediate Value Theorem** which states.

If f is continuous on $[a,b]$ then $f(x)$ takes on every value between $f(a)$ and $f(b)$.

This result should be intuitive. If we have a connected path from $f(a)$ to $f(b)$ then the path must pass through each value between the endpoints.

Things we need to know and understand:

- How to determine continuity at a point
- Consequences of Intermediate Value Theorem

Review Questions:

8. Show that $f(x) = \begin{cases} \frac{x-2}{\sqrt{x+2}-2}, & x < 2 \\ 4 \quad , & x = 2 \\ \frac{4x-8}{\sin(x^2-3x+2)}, & x > 2 \end{cases}$ is continuous at $x = 2$.
9. Construct a function that is continuous on (a, b) but does not satisfy the Intermediate Value Theorem.
10. Show that $\tan x = \ln x^2$ has a solution.

Solutions:

8. Both the one-sided limits and $f(2) = 4$. $\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+2)}{x-2} = 4$ and $\lim_{x\to 2} \frac{4x-8}{\sin(x^2-3x+2)} = \lim_{x\to 2} \frac{4(x-2)}{\sin((x-2)(x-1))} = 4 \cdot \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-1)\cdot\sin((x-2)(x-1))} = 4$
9. Consider $f(x) = floor(x)$ which is continuous on $(0,1)$ but $f(1) = 1$ and $f(0) = 0$ clearly there is no value $c \in (0,1)$ such that $f(c) \in (0,1)$ since $f(x) = 0$ for every $x \in (0,1)$. [Note: $floor(x)$ rounds down to the nearest integer.]
10. We want to show that $f(x) = \tan x - \ln x^2$ has a zero. Try some values of x to find a positive value. Upon checking $f(1) = 1.6$ and $f(2) = -3.6$. Since f is continuous on $[1,2]$ we have that there must be a zero by IVT.

Differentiability

A function is said to be **differentiable** if the slope (the derivative) exists at $x = c$. That means that the following limit exists:

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

This is the definition of the slope at $x = c$ and so you can intuit from the graph or data what the value of this limit should be by imagining a tangent line and when it doesn't exist. Instances when it doesn't exist is when there is no well-defined tangent line, or the slope of the tangent line is undefined. See our notes for complete list of cases where differentiability breaks down.

To define the derivative at any point x we need to change our label of $c = x$

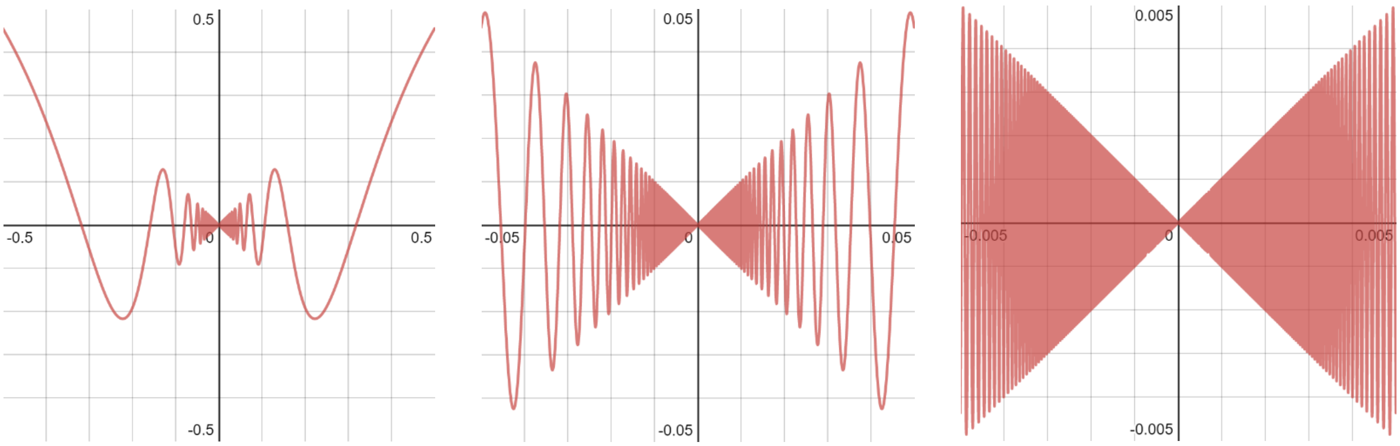
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{c\to x} \frac{f(c) - f(x)}{c - x}$$

And we denote this as $f'(x)$ or $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$. Since this is slope, we are measuring a rate of change and the $\frac{d}{dx}$ notation reflects this as it means the change of some function with respect to how much x changes. That is for a small change in x how much greater or less will the function change. We can graph this on our calculators using $\text{nDeriv}(f(x), x, x)$.

If we want to know the derivative at point, we will denote it as $f'(c)$ or $\frac{d}{dx}f(x)\Big|_{x=c}$ and if we are using our calculator can compute it using $\text{nDeriv}(f(x), x, c)$. We just need to be careful when using our calculator to test at a point that may not be differentiable. Our calculator is not actually using a limit but an approximation to the limit and will likely give incorrect results as a result. For example, $\text{nDeriv}(|x|, x, 0) = 0$ when it actually is not defined since the left and right-hand limits of the derivative at $x = 0$ are not the same.

In these cases where the function may not be differentiable you can check by zooming in and observing if the function is approaching a linear function. This is called **local linearization** and it is a property of differentiability.

For example, the function $f(x) = \begin{cases} x \cdot \sin\frac{1}{x}, x \neq 0 \\ 0, \ x = 0 \end{cases}$ is not locally linear at $x = 0$ and therefore not differentiable.



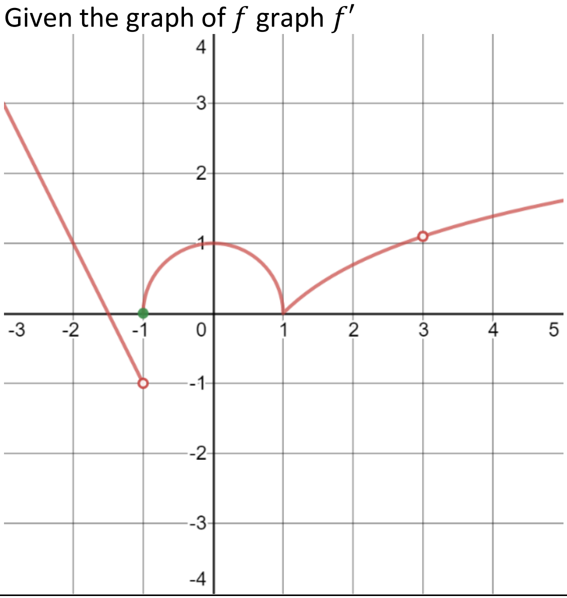
Lastly, if f is differentiable then it is also continuous. The proof is an example below, but you should be able to intuit that if $\lim_{x \to c} f(x) \neq f(c)$ then either there is a jump or asymptote which cannot have a real valued tangent slope, or there is a hole and we will need to use the value of $f(c)$ which is different than the neighbourhood around c.

Things we need to know and understand:

- How to find the slope at a point using limits and a calculator
- How to find the graph of f' given the graph or data of f, and can graph the derivative on a calculator using the original
- How to determine differentiability using limits and the shape of the function
- That differentiable functions are locally linear and continuous

Review Questions:

11. Determine y' if $y = \frac{x}{\sqrt{x^2+1}}$ and state where it is differentiable. Compute nDeriv where it is not differentiable.
12. Determine $\frac{d}{dx}|x^2 - 1|$ and state where it is differentiable. Compute nDeriv where it is not differentiable.
13. Given the graph of f graph f'



14. Given the table of f graph f' . Assume f is continuous

x	2	4	8	11	12	15	20	21	25
$f(x)$	-2	6	10	1	-5	1	6	9	11

15. Prove that differentiability implies continuity.

Solutions:

11. $y' = \lim_{c\to x} \frac{\frac{c}{\sqrt{c^2+1}}-\frac{x}{\sqrt{x^2+1}}}{c-x} = \lim_{c\to x} \frac{c\sqrt{x^2+1}-x\sqrt{c^2+1}}{(c-x)\sqrt{c^2+1}\cdot\sqrt{x^2+1}} \cdot \frac{c\sqrt{x^2+1}+x\sqrt{c^2+1}}{c\sqrt{x^2+1}+x\sqrt{c^2+1}} = \lim_{c\to x} \frac{c^2(x^2+1)-x^2(c^2+1)}{(c-x)\sqrt{c^2+1}\cdot\sqrt{x^2+1}(c\sqrt{x^2+1}+x\sqrt{c^2+1})}$

$y' = \lim_{c\to x} \frac{c^2-x^2}{(c-x)\sqrt{c^2+1}\cdot\sqrt{x^2+1}(c\sqrt{x^2+1}+x\sqrt{c^2+1})} = \lim_{c\to x} \frac{c+x}{\sqrt{c^2+1}\cdot\sqrt{x^2+1}(c\sqrt{x^2+1}+x\sqrt{c^2+1})} = \frac{2x}{(x^2+1)(2x\sqrt{x^2+1})} = (x^2+1)^{-\frac{3}{2}}$

Since $x^2+1>0$ always, y' does not have any discontinuity and therefore y' exists $\forall\ x$ and is differentiable everywhere.

12. $|x^2-1| = \begin{cases} x^2-1, & x\le -1 \text{ or } x\ge 1 \\ 1-x^2, & -1<x<1 \end{cases}$ so, we need to consider two cases.

Case 1: $|x|\ge 1$ then

$$\frac{d}{dx}|x^2-1| = \lim_{h\to 0}\frac{(x+h)^2-1-(x^2-1)}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = 2x$$

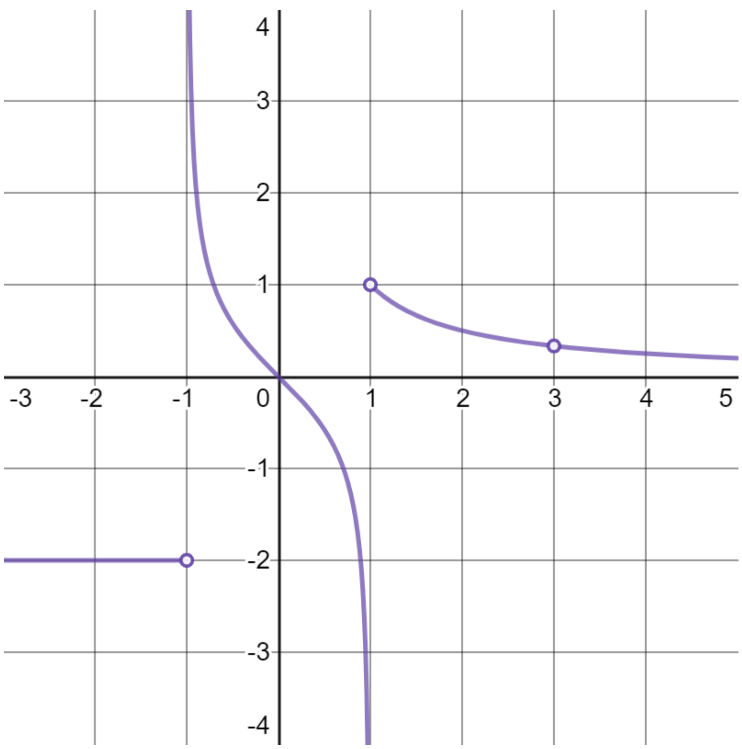
Case 2: $|x|<1$ then

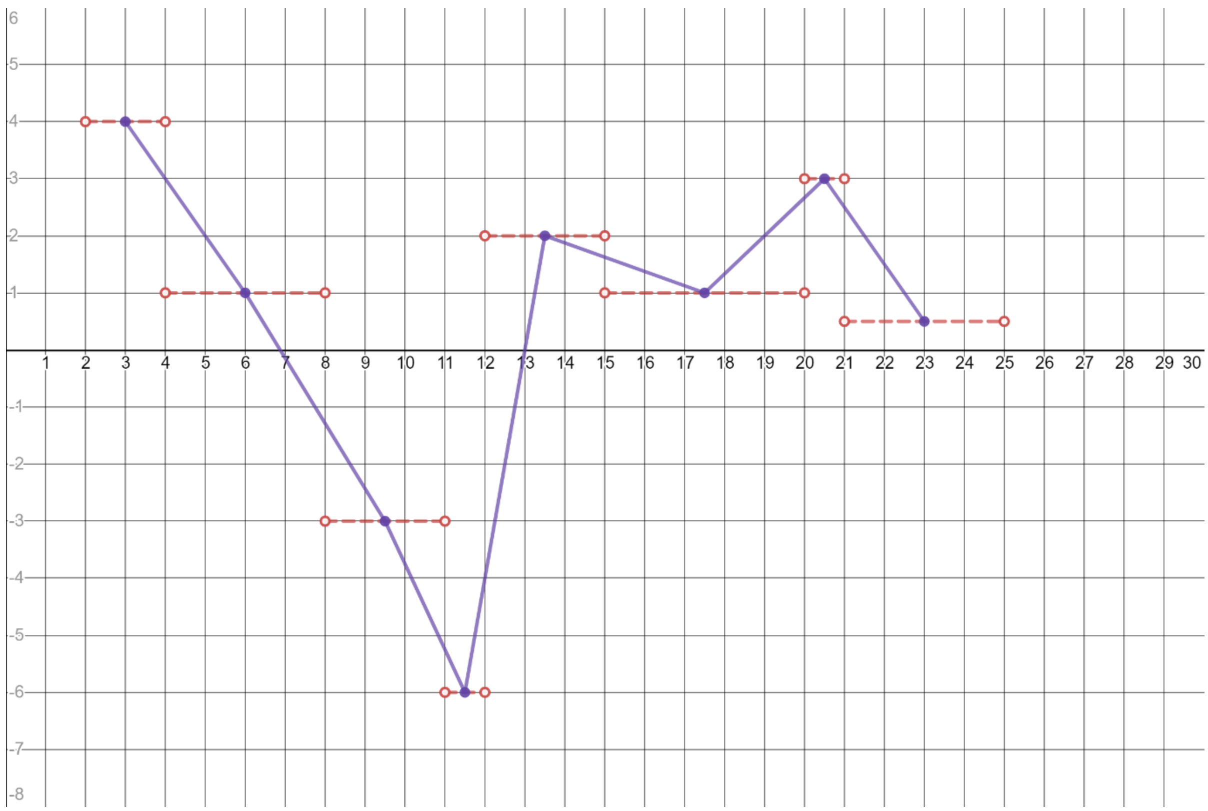
$$\frac{d}{dx}|x^2-1| = \lim_{h\to 0}\frac{1-(x+h)^2-(1-x^2)}{h} = \lim_{h\to 0}\frac{-2xh-h^2}{h} = -2x$$

Hence, if we wanted to evaluate the derivative at ± 1 we would get ± 2 depending the side of ± 1 we approach, and the limit definition would not exist. So $|x^2-1|$ is differentiable $\forall\ x\in\mathbb{R}, x\ne\pm 1$

nDeriv$(|x^2-1|, x, -1) = -0.001$ and nDeriv$(|x^2-1|, x, 1) = 0.001$

13.



14. 

The dotted horizontal lines assume piecewise linear, the solid line is connecting the midpoints of each region with linear equations.

15. The textbook gives a proof that shows if f is differentiable then f is continuous. My style is to show that if f is not continuous then it is not differentiable (stop to think why this is an equivalent statement).

So, if f is not continuous at $x = c$ then maybe $f(c)$ is not defined. In that case $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ cannot be evaluated since it requires that $f(c)$ be something. If we have that $f(c)$ exists, then either $\lim_{x \to c} f(x)$ does not exist or $\lim_{x \to c} f(x) \neq f(c)$. In the both cases, we are going to break the definition of a limit.

We want that $\forall\ \varepsilon > 0, \exists\ \delta > 0$ such that if $|x - c| < \delta$ then $\left|\frac{f(x)-f(c)}{x-c}\right| < \varepsilon$. In the first case where the limit of f does not exist, we have no hope for $\left|\frac{f(x)-f(c)}{x-c}\right| < \varepsilon$ since this requires us to bound $f(x)$ around $x = c$.

$$\left|\frac{f(x)-f(c)}{x-c}\right| > \frac{|f(x)-f(c)|}{\delta}$$

And we have no way to bound f around anything. If instead, the limit of f exists but is not $f(c)$ then we let

$$\Delta = |f(x) - f(c)| > 0$$

Our inequality will become

$$\left|\frac{f(x)-f(c)}{x-c}\right| > \frac{|f(x)-f(c)|}{\delta} > \frac{\Delta}{\delta}$$

Which is fixed and making δ smaller only makes the lower bound greater so this will never be smaller than an arbitrarily small ε ■

You could use a similar argument starting with differentiability and building $|f(x) - f(c)| < \varepsilon$ for $\delta = \varepsilon_1 \delta_1$ where $|x - c| < \delta_1 \Rightarrow \left|\frac{f(x)-f(c)}{x-c}\right| < \varepsilon_1$ in a much more elegant argument, but I am partial to arguments that use negations.