The Basics: Power, Sum, Product, Quotient, Chain

We need to know how to derive these rules and how to use them in context. All of these rules will satisfy the definition of the derivative, but this is not always the best way to illustrate why they are true. The first two limits work nicely and the other 3 we need a visual representation to help. The limit definition will use the expression below:

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

To visualize product and quotient rule using an area model is quite powerful and chain rule works by considering how rates multiply together and units cancel and then takings a limit as the rate goes to 0/0. I leave the method of showing why these rules are true to you below, but the five rules are:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\big(f(x)+g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}(f\cdot g) = g\cdot\frac{df}{dx} + f\cdot\frac{dg}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\cdot\frac{df}{dx} - f\cdot\frac{dg}{dx}}{g^2}$$

$$\frac{d}{dx}f(u) = \frac{df}{du}\cdot\frac{du}{dx}$$

Remember that derivative is slope (rate of change) so these derivates will be used to measure rates of change.

Example: Find $\frac{dy}{dz}$ of the following:

$$y = \sqrt{\frac{xw}{\sqrt{z^2+2z}}}$$

Solution: You should be able to describe the plan of attack as chain rule; then quotient rule which will use product rule and chain rule inside it; power rule and sum rule are clearly there but they will always be when we have a polynomial. Also, there will be dx/dz and dw/dz because of chain rule.

$$\frac{dy}{dz} = \frac{1}{2}\sqrt{\frac{\sqrt{z^2+2z}}{xw}}\cdot\frac{\left(w\cdot\frac{dx}{dz} + x\cdot\frac{dw}{dz}\right)\sqrt{z^2+2z} - \frac{(2z+2)}{2\sqrt{z^2+2z}}\cdot xw}{|z^2+2z|}$$

I would not bother to simplify this.

Things we need to know and understand:

- How to justify the five basic derivative rules
- How to take the derivative of any polynomial function and use the trig rules (especially with chain rule!)
- How to recognize limits as derivatives

Review Questions:

1. Prove Power Rule
2. Prove Sum Rule
3. Illustrate Product Rule
4. Illustrate Quotient Rule
5. Justify Chain Rule
6. Find dy/dx of $y = \sqrt[4]{\frac{2x(x+1)}{5x^3-4}}$

Solutions:

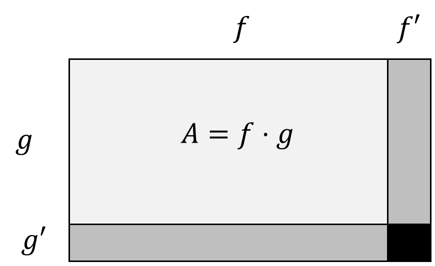
1. Use limits

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n + nx^{n-1}h + \sigma(h^2) - x^n}{h}$$
$$= \lim_{h\to 0}\frac{nx^{n-1}h + \sigma(h^2)}{h} = \lim_{h\to 0} nx^{n-1} + \sigma(h) = nx^{n-1}$$

2. Use limits

$$\frac{d}{dx}(f+g) = \lim_{h\to 0}\frac{\big(f(x+h)+g(x+h)\big) - \big(f(x)g+(x)\big)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \frac{dg}{dx}+\frac{df}{dx}$$

3. Use an area model

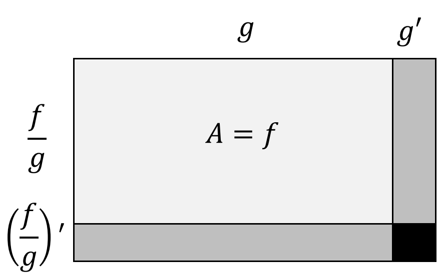


$$A' = (f\cdot g)' = f'g + g'f + g'f'$$

But $g'f' \ll g'f$ and is essentially a point once the change in f and g go to 0 so it is treated as 0.

$$(f\cdot g)' = f'g + g'f$$

4. Use the same area model but different labels



$$A' = f' = g'\cdot\frac{f}{g} + \left(\frac{f}{g}\right)' g$$

And we need to solve for $(f/g)'$

$$g^2\cdot\left(\frac{f}{g}\right)' = f'g - g'f$$
$$\left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2}$$

5. We know that

$$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$

In the discrete and large world. We also know

$$\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \frac{df}{dx} = \lim_{\substack{\Delta x\to 0\\ \Delta u\to 0}}\frac{\Delta f}{\Delta u}\cdot\frac{\Delta u}{\Delta x} = \frac{df}{du}\cdot\frac{du}{dx}$$

6.

$$\frac{dy}{dx} = \frac{1}{4}\cdot\left(\frac{2x(x+1)}{5x^3-4}\right)^{-\frac{3}{4}}\cdot\frac{(4x+2)(5x^3-4)-(15x^2)(2x^2+2x)}{(5x^3-4)^2}$$

Special Functions: Trig and Exponential

We used the sum of angle identities to find the derivative of sine and cosine and were able to determine the derivative for each trig function below using quotient rule.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cdot \cot x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

The exponential function was built to have the property that it is its own derivative. You don't need to know the series of e, but you should feel comfortable that e was built to solve a certain problem.

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}b^x = b^x \cdot \ln b$$

Things we need to know and understand:

- How to use sum of angles to show the derivative of sine and cosine are each other.
- Why the derivative of sine and cosine require radians.
- How to take the derivative of all 6 trig ratios.
- That e was constructed to satisfy the equation $y' = y$
- How to take derivative of b^x
- The couple limits involving e

Review Questions:

7. Prove $\frac{d}{dx}\sin x = \cos x$
8. Show that $\frac{d}{dx}\sec x = \sec x \tan x$
9. Why do trig derivatives need to be measured in radians?
10. Is the derivative of
$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
Continuous at $x = 0$? How about $g(x) = xf(x)$?
11. What is special about e^x in terms of where it came from or how it was derived?
12. Find dy/dx given that
$$y = \tan e^{u^2}$$
13. Prove $\frac{d}{dx}b^x = b^x \cdot \ln b$
14. What is the value of the following limit?
$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = L$$
15. What is the value of the following limit?
$$\lim_{h\to 0}\frac{e^h - 1}{h} = L$$

Solutions:

7. Use limits

$$\begin{aligned}\lim_{h\to0}\frac{\sin(x+h)-\sin x}{h} &= \lim_{h\to0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}\\ &= \lim_{h\to0}\frac{\sin x\,(\cos h-1)-\sin h\cos x}{h}\\ &= \lim_{h\to0}\frac{\sin x\,(\cos h-1)}{h}-\lim_{h\to0}\frac{\sin h\cos x}{h}\\ &= \sin x\lim_{h\to0}\frac{(\cos h-1)}{h}-\cos x\lim_{h\to0}\frac{\sin h}{h}=\sin x\cdot0-\cos x\cdot1=-\cos x\end{aligned}$$

8. Use quotient rule

$$\frac{d}{dx}\sec x=\frac{d}{dx}\frac{1}{\cos x}=\frac{(0+\sin x)}{\cos^2 x}=\tan x\cdot\sec x$$

9. Because they use the limit

$$\lim_{h\to0}\frac{\sin h}{h}=1$$

Which is only true in radians thanks to Sandwich Theorem.

10.

$$f'(x)=\begin{cases}2x\sin(1/x)+\cos(1/x), & x\neq0\\ 0, & x=0\end{cases}$$

We get that $f'(0)=0$ by using the limit definition

$$\begin{aligned}f'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}&=\lim_{h\to0}\frac{h^2\sin(1/h)-0}{h}\\ &=\lim_{h\to0}h\sin(1/h)=0\end{aligned}$$

The function is not continuous since there is an oscillating discontinuity at $x=0$ for $\cos(1/x)$, hence $\lim_{x\to0}f(x)$ does not exist. However, g is continuous because

$$g'(x)=\begin{cases}3x^2\sin(1/x)+x\cos(1/x), & x\neq0\\ 0, & x=0\end{cases}$$

Which does have the limit value go to 0 as x goes to 0

11. $y=e^x$ was intentionally constructed to solve that equation. Nothing too fancy about it.

12.

$$\frac{dy}{dx}=\sec^2\left(e^{u^2}\right)\cdot e^{u^2}\cdot2u\cdot\frac{du}{dx}$$

13. Use derivative rules. If we have that

$$y=b^x$$

Then for some value of k, we have that $b=e^k$. So, we want the derivative of $y=\left(e^k\right)^x$

$$\begin{aligned}\frac{dy}{dx}=\frac{d}{dx}\left(e^k\right)^x&=\frac{d}{dx}e^{kx}\\ &=e^{kx}\cdot k\end{aligned}$$

Using the fact that $\frac{d}{dx}e^u=e^u\cdot\frac{du}{dx}$. Then we can see that $k=\ln b$ and $e^{kx}=b^x$

$$\frac{dy}{dx}=b^x\cdot\ln b$$

14. The limit comes from compound interest and does not require more explanation.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

15. The limit comes from the definition of the derivative of e^x and does not require more explanation.

$$\lim_{h\to0}\frac{e^h-1}{h}=1$$

Derivative Tricks: Implicit and Logarithmic

We can use chain rule to take the derivative of inverse functions by doing algebra to both sides. For inverse trig and logarithmic functions, the big idea is that

$$y = f(x) \Rightarrow x = f^{-1}(y) = g(y)$$
$$\frac{d}{dx}x = \frac{d}{dx}f^{-1}(y) = \frac{d}{dx}g(y)$$
$$1 = g'(y) \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{g'(y)} = \frac{1}{g'\big(f(x)\big)}$$

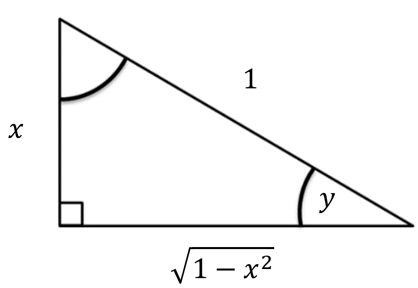
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \frac{d}{dx}\log_{\mathrm{b}} x = \frac{1}{x \cdot \ln b}$$

To prove what $g'\big(f(x)\big)$ is for the inverse trig functions use the geometry on a triangle and your knowledge of what the trig ratios represent.



So, with this diagram we can simplify $\cos(\arcsin x)$.

We know that $y = \arcsin x$ and that x is the ratio opposite over hypotenuse and y is the angle. Once that is identified, we can see that $\cos y = \sqrt{1-x^2}$

We also discussed the strategy of taking the natural log of both sides and then differentiating as a short cut when finding complex exponential derivatives and saving time with quotient rule and product rule thanks to log laws.

Things we need to know and understand:

- How to take the derivative of an inverse function using chain rule
- How to take the derivative logarithmically

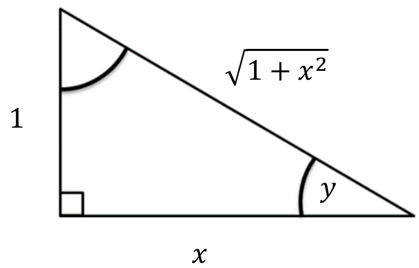
Review Questions:

16. Show that $\frac{d}{dx}\operatorname{arccot} x = -1/(1+x^2)$
17. Find $\frac{dy}{dx}$ of $y = \sqrt[x]{x}$
18. Find $\frac{d^2y}{dx^2}$ of $x^2 - y^3 = y$
19. Derive power rule, product rule, and quotient rule using logarithms.
20. Find $\frac{dy}{dx}$ for $y = \left(\frac{(x+1)(z-1)}{(w-2)(x+3)}\right)^5$

Solutions:

16. If $y = \operatorname{arccot} x$ then $x = \cot y$ so

$$\frac{d}{dx}x = \frac{d}{dx}\cot y$$
$$1 = -\csc^2 y \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = -\sin^2 y = -\frac{1}{\left(\sqrt{1+x^2}\right)^2} = -\frac{1}{|1+x^2|} = -\frac{1}{1+x^2}$$



17. If $y = \sqrt[x]{x}$ then $\ln y = \ln x^{\frac{1}{x}} = \frac{1}{x} \cdot \ln x$

$$\frac{1}{y} \cdot \frac{dy}{dx} = -\frac{1}{x^2}\ln x + \frac{1}{x^2}$$
$$\frac{dy}{dx} = \frac{y}{x^2}(1 - \ln x) = \frac{\sqrt[x]{x}}{x^2}(1 - \ln x)$$

18. Find first derivative.

$$2x - 3y^2 \cdot \frac{dy}{dx} = \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x}{1+3y^2}$$

Then the second.

$$\frac{d^2y}{dx^2} = \frac{2(1+3y^2) - 12yx \cdot \frac{dy}{dx}}{(1+3y^2)^2}$$

You could simplify this by substituting for dy/dx

19. If $y = x^n$ then $\ln y = n \cdot \ln x$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{n}{x} \Rightarrow \frac{dy}{dx} = y \cdot \frac{n}{x} = \frac{nx^n}{x} = nx^{n-1}$$

If $y = f \cdot g$ then $\ln y = \ln f + \ln g$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{f} \cdot \frac{df}{dx} + \frac{1}{g} \cdot \frac{dg}{dx}$$
$$\frac{dy}{dx} = y\left(\frac{1}{f} \cdot \frac{df}{dx} + \frac{1}{g} \cdot \frac{dg}{dx}\right) = fg\left(\frac{1}{f} \cdot \frac{df}{dx} + \frac{1}{g} \cdot \frac{dg}{dx}\right) = g \cdot \frac{df}{dx} + f \cdot \frac{dg}{dx}$$

If $y = f/g$ then $\ln y = \ln f - \ln g$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{f} \cdot \frac{df}{dx} - \frac{1}{g} \cdot \frac{dg}{dx}$$
$$\frac{dy}{dx} = y\left(\frac{1}{f} \cdot \frac{df}{dx} - \frac{1}{g} \cdot \frac{dg}{dx}\right) = \frac{f}{g}\left(\frac{1}{f} \cdot \frac{df}{dx} - \frac{1}{g} \cdot \frac{dg}{dx}\right) = \frac{1}{g} \cdot \frac{df}{dx} - \frac{f}{g^2} \cdot \frac{dg}{dx}$$

20. Using logs we get

$$\ln y = 5(\ln(x+1) + \ln(z-1) - \ln(w-2) - \ln(x+3))$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = 5\left(\frac{1}{x+1} + \frac{1}{z-1} \cdot \frac{dz}{dx} - \frac{1}{w-2} \cdot \frac{dw}{dx} - \frac{1}{x+3}\right)$$
$$\frac{dy}{dx} = 5\left(\frac{(x+1)(z-1)}{(w-2)(x+3)}\right)^5\left(\frac{1}{x+1} + \frac{1}{z-1} \cdot \frac{dz}{dx} - \frac{1}{w-2} \cdot \frac{dw}{dx} - \frac{1}{x+3}\right)$$