Approximating the Area Under a Curve

Given a function f, we said that we could approximate the signed area under the curve on the interval $[a, b]$ if we used a bunch of rectangles. In general, the idea is to partition the interval into n pieces so we have

$$P = \{x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b\}$$

And the length of each subinterval is $\Delta x_k = x_k - x_{k-1}$ and we can pick some $c_k \in [x_{k-1}, x_k]$ so that $f(c_k)$ is the height. Adding up all these areas we get a Riemann Sum

$$\sum_{k=1}^{n} f(c_k)\Delta x_k \approx \text{Area}$$

The Riemann sum will become a better approximation as the partition as it becomes finer (the size of the longest interval (norm) goes to 0, $\|P\| \to 0$)

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \text{Area} \stackrel{\text{def}}{=} \int_a^b f(x)dx$$

If we want to use a regular partition so that $\Delta x_k = \frac{b-a}{n}$ $\forall\, k$ then we have three options for our choice of $c_k \in [x_{k-1}, x_k]$

The right endpoint, $c_k = x_k$, the left endpoint, $c_k = x_{k-1} = x_k - \Delta x$, or the middle point, $c_k = \frac{x_k + x_{k-1}}{2} = x_k - \frac{1}{2}\Delta x$

In general, $c_k = x_k - T\Delta x$, where $T = 0$ for RRAM, $T = 1$ for LRAM and $T = 0.5$ for MRAM (as in our program), but for this example I will use RRAM since it simplifies the expression. The only thing left to do is note $x_k = a + k \cdot \Delta x$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} = \text{Area}$$

Example: Write the following integral as a Riemann sum

$$\int_{-1}^{2} (x^2 + x)dx$$

Solution:

We can write it in terms of an arbitrary c_k given an arbitrary partition P, and as RRAM.

$$\int_{-1}^{2} (x^2 + x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k^2 + c_k)\Delta x_k \text{ where } P \text{ is a partiton of } [-1, 3]$$

$$\int_{-1}^{2} (x^2 + x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(\frac{3k}{n} - 1\right)^2 + \left(\frac{3k}{n} - 1\right)\right) \cdot \frac{3}{n}$$

Example: Write the following Riemann sum as an integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{4k}{n} + 2\right) \cdot \frac{2}{n}$$

Solution:

Two ideas. The first is we identify $\frac{2}{n}$ as $\frac{b-a}{n}$ and then note the interval has length 2. Then to identify the function we clearly see our $f(x) = \ln g(x)$ for some linear g. There are multiple approaches from here. We notice that $g(a) = 2$ and $g(b) = 6$ and each step is moving twice as fast as Δx. We could have $g(x) = 2x$ where $x \in [1, 3]$ or we could shift the function by making $a = 0$ so $\bar{g}(x) = 2x + 2$ and $x \in [0, 2]$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{4k}{n} + 2\right) \cdot \frac{2}{n} = \int_0^2 \ln(2x + 2)\, dx = \int_1^3 \ln(2x)\, dx$$

The second is setting $\frac{4k}{n} + 2 = k\Delta x + a$ immediately, so that $a = 2$ and $\frac{4}{n} = \frac{b-a}{n}$. This way we get that $g(x) = x$ where $x \in [2, 6]$ or we could have that $\bar{g}(x) = x + 2$ with $x \in [0, 4]$. In both cases we have $g(a) = \bar{g}(a) = 2$ and $g(b) = \bar{g}(b) = 6$. The only problem is we don't have $\frac{4}{n}$ in the sum so we multiply and divide by 2.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\ln\left(\frac{4k}{n} + 2\right) \cdot \frac{4}{n} = \int_0^4 \frac{1}{2}\ln(x + 2)\, dx = \int_2^6 \frac{1}{2}\ln(x)\, dx$$

We also used trapezoids to approximate the area. We have a regular partition $\Delta x = \frac{b-a}{n}$ and on the interval $[x_{k-1}, x_k]$ we get an average height of $\frac{1}{2}\left(f(x_{k-1}) + f(x_k)\right)$. Then the integral can be approximated as

$$\int_a^b f(x)dx \approx \left(\frac{1}{2}f(a) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right)\Delta x$$

Since we add each middle point twice.

Finally, we have a set of integral properties that we can prove from the limit definition and the area under a curve definition. One really important identity is the average value of a function on $[a, b]$

$$\operatorname{avg} f = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{f(c_k)}{n}$$

For a partition P of $[a, b]$

$$\operatorname{avg} f = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{f(c_k)}{n} \cdot \frac{\Delta x}{\Delta x} = \lim_{\|P\|\to 0} \frac{1}{n \cdot \Delta x} \sum_{k=1}^{n} f(c_k)\Delta x$$

Set $\Delta x = \frac{b-a}{n}\ \forall k$

$$\operatorname{avg} f = \lim_{\|P\|\to 0} \frac{1}{b-a} \sum_{k=1}^{n} f(c_k)\Delta x = \frac{1}{b-a} \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx$$

And we get that for continuous functions f, IVT guarantees $\exists$ some $c \in [a, b]$ such that $f(c) = \operatorname{avg} f$. This is called the Mean Value Theorem (for Integrals).

Things we need to know and understand:

- How to approximate an area using RRAM and Trapezoid method
- How to evaluate an integral using geometry and symmetry
- How to evaluate an integral using your calculator
- How to write an integral as a Riemann sum and how to write a Riemann sum as an integral
- How to find the average value of a function
- Why integral properties are true

Review Questions:

1. Find the area under the curve $f(x) = \frac{1}{x^2+1}$ on the interval $[-4, 4]$ as follows:
   a. Using 4 subintervals and MRAM
   b. Using 4 subintervals and Trapezoid Method
   c. Using your calculator
2. Write the following integral as a Riemann sum using a limit:
   a. $\|P\| \to 0$
   b. $n \to \infty$

$$\int_0^{\pi} \sin^2 x\, dx$$

3. Write the following limit as a discrete integral and then evaluate it using geometry.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{6k}{n}\right) \cdot \frac{1}{n}$$

4. Find the average distance between the curve $y = 4 - x^2$ and the origin on the interval $x \in [-2, 2]$.
5. Prove that

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Solutions:

1.
   a. MRAM we get a partition of $P = \{-4, -2, 0, 2, 4\}$ and $\Delta x = 2$
   $$\text{Area} \approx 2\big(f(-3) + f(-1) + f(1) + f(3)\big) = 2.4$$
   b. Trapezoid we get
   $$\text{Area} \approx 2\left(\frac{1}{2}f(-4) + f(-2) + f(0) + f(2) + \frac{1}{2}f(4)\right) = 2.918\ldots = \frac{248}{85}$$
   c. Using our calculator, we have
   $$\int_{-4}^{4} \frac{1}{x^2+1}\,dx = 2.651635\ldots$$
2.
   a. Let P be a partition of $[a, b]$
   $$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sin^2 c_k \,\Delta x$$
   b. Use $\Delta x_k = \frac{\pi}{n}$, so $c_k = x_k = \frac{k\pi}{n}$
   $$\lim_{n\to\infty} \sum_{k=1}^{n} \sin^2\left(\frac{\pi k}{n}\right)\cdot\frac{\pi}{n}$$
3. Multiple solutions of course, my first approach would be to set an interval of length 1 and make $f(a) = 3$ and $f(b) = 9$ for linear f. It moves at a step 6 times faster than x so $f(x) = 6x + 3$ for $x \in [0, 1]$
   $$\int_0^1 (6x+3)dx = \frac{9+3}{2}\cdot 1 = 6$$
   Since this makes a trapezoid.
4. We get the distance between the curve $y = 4 - x^2$ and the point $(0, 0)$ is $d(x) = \sqrt{x^4 - 7x^2 + 16}$
   $$\operatorname{avg} f = \frac{1}{4}\int_{-2}^{2}\sqrt{x^4 - 7x^2 + 16}dx = \frac{1}{4}(12.208\ldots) = 3.0521\ldots$$
5. We have that
   $$\int_a^b f(x)dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x_k$$
   For the partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ where $\Delta x_k = x_k - x_{k-1} > 0\ \forall\ k$. Then consider the partition $\mathbb{P} = \{b = x_n, x_{n-1}, \ldots, x_1, x_0 = a\}$ with $\Delta \mathbb{x}_k = x_{k-1} - x_k < 0\ \forall\ k$.
   $$\lim_{\|\mathbb{P}\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta\mathbb{x}_k = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta\mathbb{x}_k = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)(-\Delta x_k) = -\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

Fundamental Theorem of Calculus

We can define the area under the curve of f on the interval $[a, x]$ as a function:
$$F(x) = \int_a^x f(t)dt$$
We showed that $\Delta F(x) = f(c)\Delta x$ for some $c \in [x, x + \Delta x]$ by MVT. This is equivalent to $\frac{\Delta F(x)}{\Delta x} = f(c)$ and we can let $\Delta x \to 0$.
$$\lim_{\Delta x\to 0}\frac{\Delta F(x)}{\Delta x} = \lim_{\Delta x\to 0} f(c)$$
$$\Rightarrow \frac{dF}{dx} = f(x)$$
And so F is the antiderivative of f. That is to say, it is some function that we can differentiate and get f.

The first part of the Fundamental Theorem of Calculus states this fact:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

The second part states that since F is an antiderivative then define F as follows

$$F(x) = \int_0^x f(t)dt$$

Then

$$\begin{aligned}\int_a^b f(t)dt &= \int_a^0 f(t)dt + \int_0^b f(t)dt \\ &= -\int_0^a f(t)dt + \int_0^b f(t)dt \\ &= F(b) - F(a)\end{aligned}$$

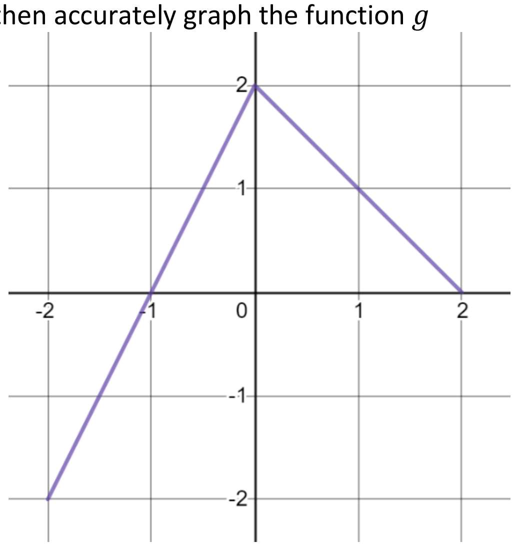
We want to be comfortable analyzing functions defined as integrals by using FTC.

Things we need to know and understand:

- How to derive the Fundamental Theorem
- How to analyze functions defined as integrals
- How to use both parts of FTC

Review Questions:

6. Using the limit definition of derivative show that
$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$
7. If h is the function shown below then accurately graph the function g



$$g(x) = \int_x^{2x} h(t)dt$$

8. Determine difference in the area between the curves $f(x) = \cos x$ and $g(x) = 1 - \frac{x^2}{2}$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Solutions:

6. With the limit definition we have

$$\frac{d}{dx}\int_a^x f(t)dt = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\cdot\int_x^{x+h} f(t)dt$$
$$= \lim_{h\to 0} f(c) \qquad \text{By the MVT we can find } c\in[x,x+h] \text{ where } f(c)=\text{avg} f$$
$$= f(x) \qquad \text{As } h\to 0 \text{ we have } c\to x$$

7. We want to identify $g'(x)$ and $g''(x)$

$$g'(x) = 2h(2x) - h(x)$$
$$g''(x) = 4h'(2x) - h'(x)$$

Since we are comparing $h(x)$ and $h(2x)$ we can write h as a piecewise function

$$h(x) = \begin{cases} 2x+2, -2\le x<0 \\ 2-x, 0\le 0\le 2\end{cases}$$

$$h(2x) = \begin{cases} 4x+2, -1\le x<0 \\ 2-2x, 0\le x<1\end{cases}$$

$$\Rightarrow g'(x) = \begin{cases} 6x+2, -1\le x<0 \\ 2-3x, 0\le x\le 1\end{cases}$$

So, g' changes sign when it passes through 0 which happens when $6x+2=0\Rightarrow x=-\frac{1}{3}$ and when $2-3x=0\Rightarrow x=\frac{2}{3}$ are minimums and maximums respectively.

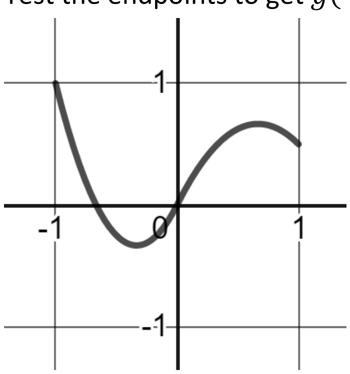
$$g\left(-\frac{1}{3}\right) = \int_{-\frac{1}{3}}^{-\frac{2}{3}} h(t)dt = -\frac{1}{3}$$

using area of a trapezoid with average height 1 and width $-\frac{1}{3}$.

$$g\left(\frac{2}{3}\right) = \int_{\frac{2}{3}}^{\frac{4}{3}} h(t)dt = \frac{2}{3}$$

There are inflection points when $g''(x)$ changes sign, which only happens at 0 where we go from positive (concave up) to negative (concave down).

Test the endpoints to get $g(-1) = \int_{-1}^{-2} h(t)dt = 1$ and $g(1) = \int_1^2 h(t)dt = \frac{1}{2}$



8. Difference in area is

$$A = \int_{-\pi/2}^{\pi/2}\left(\cos t - 1 + \frac{t^2}{2}\right)dt = \left[\sin t - t + \frac{t^3}{6}\right]_{-\pi/2}^{\pi/2} = \left(1-\frac{\pi}{2}+\frac{\pi^3}{48}\right) - \left(-1+\frac{\pi}{2}-\frac{\pi^3}{48}\right)$$
$$A = 2-\pi+\frac{\pi^3}{24}\approx 0.15$$