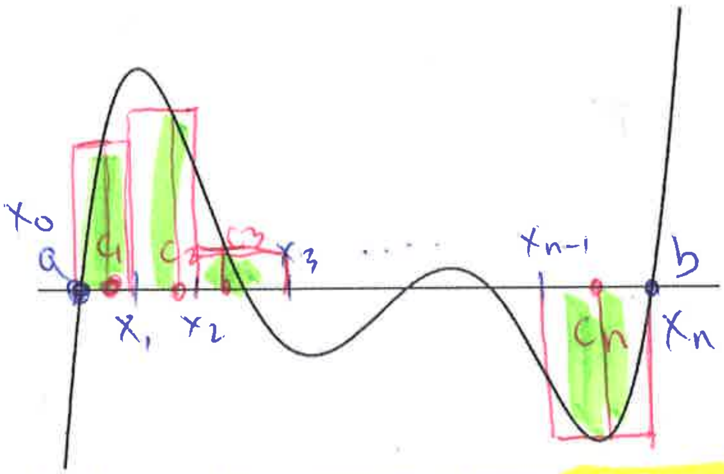
# The Definite Integral

**Goal:**
- Can write the limit of a Riemann Sum as a definite integral
- Can evaluate definite integrals using signed area under the curve
- Can use calculator to evaluate definite integrals

**Terminology:**
- Integral

**Reminder:**
- Quiz Wednesday January 15th

**Riemann Sums:**




Consider the function $f$ and we want to find the net area under the curve on $[a, b]$. In general what we can do is consider some **partition** of $[a, b]$. That is, divide the interval intosome sequence

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$$

where $x_0 = a$ and $x_n = b$ and $x_k < x_{k+1}$.

With that our job is just to estimate the area under the curve on the subinterval $[x_k, x_{k+1}]$. If the partition is uniformaly small enough (that is the largest subinterval has small length) then any regtangle in the subintervals will be a good approximation to the area.

So pick some arbitrary point in the subinterval, $c_k \in [x_k, x_{k+1}]$ and use $f(c_k)$ to make the height of the rectangle which has width $\Delta x_k = x_{k+1} - x_k$, hence it has area of $\Delta x_k \cdot f(c_k)$

in for loop
$S + DW \to S$
$\Delta x \; F(C)$

The net area will be

stop

add up everything inside summation →

$$\sum_{k=1}^{n} \left( f(c_k) \cdot \Delta x_k \right)$$

start

→ a bunch of times → start $n=1$, $k=2$, $k=3$, ... $k=n$

This is a general Riemann Sum. Observe that RAM uses $\Delta x = \frac{b-a}{n}$ and $x_{k+1} = x_k + \Delta x$ for each $k$. The differences in LRAM, MRAM, and RRAM come from the choice of $c_k$.

For this sum to be the exact area under the curve we would need the largest length to go to zero, that is $\|P\| \to 0$ (norm)

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \text{Exact Area} = \int_a^b f(x)\,dx$$

with respect to changing

integral of f(x) from a to b

This is the definite integral. Just like the slope at a point was defined using limits, the area between curves also uses limits. This is calculus in its essence: How can we measure change at an instant? How can adding 0 infinitely often give us real number?

**Example**: Evaluate the following

$$L = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{\sqrt{16 - c_k^2} \cdot \Delta x_k}_{f(c_k)\Delta x_k} + \lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{\left(\frac{c_k}{2} + 1\right) \cdot \Delta x_k}_{g(c_k)\Delta x_k}$$

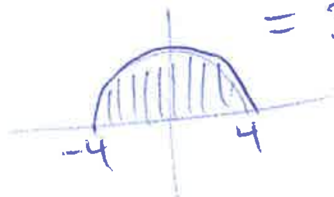
Where $P$ is any arbitrary partition of $[-4, 4]$

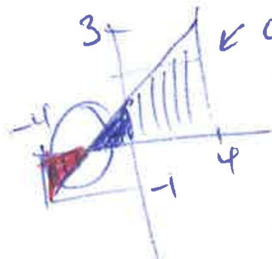
$$L = \int_{-4}^{4} \sqrt{16 - x^2}\, dx + \int_{-4}^{4} \left(\frac{x}{2} + 1\right) dx$$

find area under $\frac{x}{2}+1$ on $[-4,4]$

$$= 8\pi + 8$$

$$= 33.1327\ldots$$




avg height x width

$2 \times 4 = 8$

circle $x^2 + y^2 = r^2$

$y = \pm\sqrt{r^2 - x^2}$

Check on calculator! Using

$$\text{fnInt}(y_1, x, a, b)$$

OR on the graph you can plot the definite integral using "Calc" and entering the left and right bounds.

**Example**: Evaluate the following

$$L = \lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \cdot \sin(c_k) \cdot \Delta x_k$$

Where $P$ is some partition of $[-1, 1]$


$$L = \int_{-1}^{1} x^2 \sin x\, dx$$

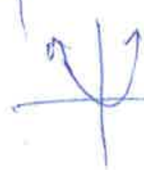
$\sin x \approx x$

$f(x) = x^2 \sin x$

$f(-x) = x^2 \sin(-x)$

$= -x^2 \sin(x)$

odd!




★ odd iff $f(x) = -f(-x)$

★ even iff $f(x) = f(-x)$

$= 0$ b/c the function (integrand) is odd and partition is symmetric about y-axis

**Practice Problems**: 5.2 # 1-6 (and change to a closed form Riemann sum), 29-40

Pick and choose out of the following sets: 7-12, 13-22, 23-28

Extra Riemann Sums

**Look Ahead:** 5.3 How can we define the average value of the function?

# Extra Riemann Sums (Hard)

For each of the following, write it as a definite integral and then estimate using MRAM and Trapezoid method ($n = 4$) and the exact using fnInt. Recall that $\Delta x = (b - a)/n = \Delta x_k$ for all $k$ in a regular partition.

1.

length

$$\int_0^3 (x^2 + x - 2)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left( \left(\frac{3k}{n}\right)^2 + \left(\frac{3k}{n}\right) - 2 \right) \cdot \frac{3}{n}$$

$$= \lim_{\|P\| \to 0} \sum_{k=1}^{n} (c_k^2 + c_k - 2)\Delta x_k$$

☆ Know

$x_k = a + k\Delta x$

$(u = a + (k-1)d)$

$\Delta x = \frac{\#}{n} = \frac{3}{n}$

$x_k = a + \frac{k\#}{n}$

$= a + \frac{3k}{n}$

2.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{2k}{n} + 1\right) \cdot \frac{2}{n}$$

3.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin^2\left(\frac{\pi k}{n}\right)}{n}$$

4.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{20e^{-\left(\frac{10k}{n}-5\right)^2}}{\sqrt{\pi}\cdot n}$$

**Solutions to integrands and intervals (Note the solutions are NOT unique!)**

1. $f(x) = x^2 + x - 2$ on $[0,3]$
2. $f(x) = \ln x$ on $[1,3]$ OR $f(x) = \ln(x+1)$ on $[0,2]$
3. $f(x) = \sin^2(\pi x)$ on $[0,1]$
4. $f(x) = \frac{e^{-\left(\frac{x}{2}\right)^2}}{\sqrt{\pi}}$ on $[-10,10]$ OR $f(x) = \frac{2e^{-x^2}}{\sqrt{\pi}}$ on $[-5,5]$