The Definite Integral

Goal:
- Can write the limit of a Riemann Sum as a definite integral
- Can evaluate definite integrals using signed area under the curve
- Can use calculator to evaluate definite integrals

Terminology:
- Integral

Reminder:
- Quiz Wednesday January 15th

Riemann Sums:

Consider the function \( f \) and we want to find the net area under the curve on \([a, b]\). In general what we can do is consider some partition of \([a, b]\). That is, divide the interval into some sequence

\[
P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}
\]

where \( x_0 = a \) and \( x_n = b \) and \( x_k < x_{k+1} \).

With that our job is just to estimate the area under the curve on the subinterval \([x_k, x_{k+1}]\). If the partition is uniformly small enough (that is the largest subinterval has small length) then any rectangle in the subintervals will be a good approximation to the area.

So pick some arbitrary point in the subinterval, \( c_k \in [x_k, x_{k+1}] \) and use \( f(c_k) \) to make the height of the rectangle which has width \( \Delta x_k = x_{k+1} - x_k \), hence it has area of

\[
\Delta x_k \cdot f(c_k)
\]

The net area will be

\[
\sum_{k=1}^{n} \left( f(c_k) \cdot \Delta x_k \right)
\]

This is a general Riemann Sum. Observe that RAM uses \( \Delta x = \frac{b-a}{n} \) and \( x_{k+1} = x_k + \Delta x \) for each \( k \). The differences in LRAM, MRAM, and RRAM come from the choice of \( c_k \).

For this sum to be the exact area under the curve we would need the largest length to go to zero, that is \( \|P\| \to 0 \)

\[
\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = \text{Exact Area} = \int_{a}^{b} f(x) \, dx
\]

This is the definite integral. Just like the slope at a point was defined using limits, the area between curves also uses limits. This is calculus in its essence: How can we measure change at an instant? How can adding 0 infinitely often give us a real number?
Example: Evaluate the following

\[ L = \lim_{||P|| \to 0} \sum_{k=1}^{n} \sqrt{16 - c_{k}^2} \cdot \Delta x_{k} + \lim_{||P|| \to 0} \sum_{k=1}^{n} \left( \frac{x_{k} + 1}{2} \right) \cdot \Delta x_{k} \]

Find area under \( f(x) \) or \([-4, 4] \)

\[ \begin{align*}
L &= \int_{-4}^{4} \sqrt{16 - x^2} \, dx \\
&= 8\pi + 8 \\
&= 33.1327
\end{align*} \]

Check on calculator! Using \( \text{fnInt}(y_{1}, x, a, b) \)

OR on the graph you can plot the definite integral using “Calc” and entering the left and right bounds.

Example: Evaluate the following

\[ L = \lim_{||P|| \to 0} \sum_{k=1}^{n} c_{k}^2 \cdot \sin(c_{k}) \cdot \Delta x_{k} \]

Where \( P \) is some partition of \([-1, 1] \)

\[ L = \int_{-1}^{1} x^2 \sin x \, dx \]

\[ \sin x \approx x \]

\[ 0 \text{ b/c the function (integrand) is odd and partition is symmetric about } y- \text{axis} \]

\[ f(x) = x^2 \sin x \]

\[ f(-x) = x^2 \sin(-x) = -x^2 \sin(x) \]

Odd \( \leftrightarrow \) odd \( \leftrightarrow \) odd \( \leftrightarrow \) odd \( \leftrightarrow \) even \( \leftrightarrow \) even

Practice Problems: 5.2 # 1-6 (and change to a closed form Riemann sum), 29-40

Pick and choose out of the following sets: 7-12, 13-22, 23-28

\( \text{Extra Riemann Sums} \)

Look Ahead: 5.3 How can we define the average value of the function?
Extra Riemann Sums (Hard)

For each of the following, write it as a definite integral and then estimate using MRAM and Trapezoid method \((n = 4)\) and the exact using \(\text{fnInt}\). Recall that \(\Delta x = (b - a)/n = \Delta x_k\) for all \(k\) in a regular partition.

1. 
\[
\int_{0}^{3} (x^2 + x - 2) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \left( \frac{3k}{n} \right)^2 + \frac{3k}{n} - 2 \right) \cdot \frac{3}{n}
\]
\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{c_k^2 + c_k - 2}{n} \right) \Delta x_k
\]
\[
\Delta x = \frac{b - a}{n} = \frac{3}{n}
\]
\[
x_k = a + \frac{k}{n}
\]
\[
= a + \frac{3k}{n}
\]

2. 
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \ln \left( \frac{2k}{n} + 1 \right) \cdot \frac{2}{n}
\]
3. \[ \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\sin^2 \left( \frac{\pi k}{n} \right)}{n} \]

4. \[ \lim_{n \to \infty} \sum_{k=1}^{n} \frac{20e^{-\left( \frac{10k}{n} - s \right)^2}}{\sqrt{n} \cdot n} \]

Solutions to integrands and intervals (Note the solutions are NOT unique!)

1. \( f(x) = x^2 + x - 2 \) on \([0, 3]\)
2. \( f(x) = \ln x \) on \([1, 3]\) OR \( f(x) = \ln(x + 1) \) on \([0, 2]\)
3. \( f(x) = \sin^2(\pi x) \) on \([0, 1]\)
4. \( f(x) = \frac{e^{-\left( \frac{10}{5} \right)^2}}{\sqrt{5}} \) on \([-10, 10]\) OR \( f(x) = \frac{2e^{-x^2}}{\sqrt{\pi}} \) on \([-5, 5]\)