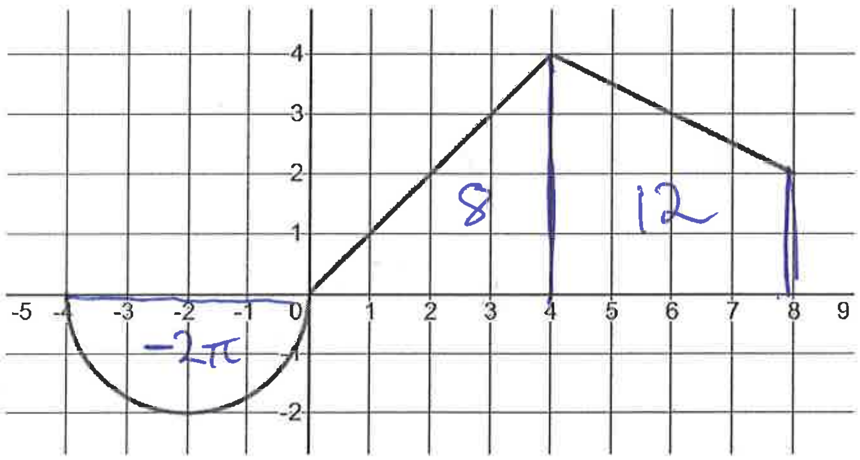
Fundamental Theorem of Calculus: Part 1

Goal: • Understands why the integral is the antiderivative • Understands why the derivative of an integral is the integrand • Can analyze functions defined as integrals
Terminology: • Antiderivative • Fundamental Theorem of Calculus

Review: Find the average value of f below on the interval $[-4, 8]$



$$\text{avg } f = \frac{1}{12}\int_{-4}^{8} f(x)\,dx$$

$$= \frac{1}{12}(20 - 2\pi)$$

$$= \frac{10 - \pi}{6}$$

$$= 1.143\ldots$$

F is a function of t

Consider the function

$$F(t) = \int_{-4}^{t} f(x)dx \leftarrow \text{area of } f \text{ on } [-4, t]$$

Where f is given above. Determine the following values: $F(-4)$, $F(0)$, $F(4)$, $F(8)$

$$F(-4) = \int_{-4}^{-4} f(x)dx = 0$$

$$F(0) = -2\pi = -6.28\ldots$$

$$F(4) = 8 - 2\pi = 1.7168.$$

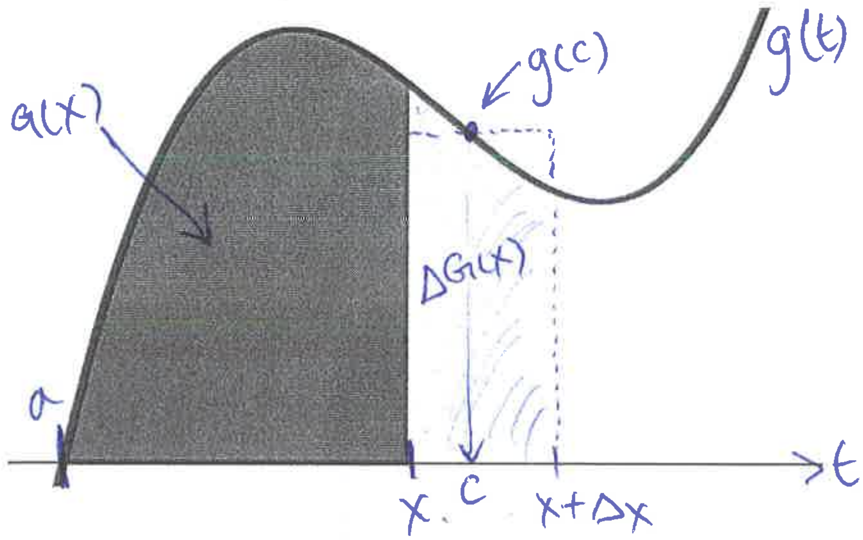
$$F(8) = 20 - 2\pi = 13.72\ldots$$

x is a bound

In general, if we have some function, g, and define a new function

$$G(x) = \int_a^x g(t)dt$$

We can consider what happens when we have a small change in x, and then consider what happens when $\Delta x \to 0$.



by MVT (of integration) we have $\exists\, c \in [x, x+\Delta x]$ s.t

$$g(c)\cdot \Delta x = \int_x^{x+\Delta x} g(t)dt = \Delta G(x)$$

want $\lim\limits_{\Delta x \to 0} \left[\Delta G(x) = g(c)\Delta x\right]$ $\Rightarrow$ $\dfrac{d}{dx}\underbrace{\boxed{\overset{G(x)}{?}}}_{\int_a^x g(t)dt} = g(x)$

$$\lim_{\Delta x \to 0} \frac{\Delta G(x)}{\Delta x} = \lim_{\Delta x \to 0} g(c)$$

$$\frac{dG}{dx} = g(x)$$

Integral is the antiderivative

$$\frac{d}{dx}\boxed{x^2} = 2x$$

$$\int_a^x 2t\,dt = x^2$$

This leads us to the first part of **Fundamental Theorem of Calculus:**

$$\frac{d}{dx}\int_a^x g(t)dt = g(x)$$

Example: Determine $h'(2)$, given that

x is in bound.

integrate wrt z

$$h(x) = \int_0^{x^2} \sin z \, dz$$

$$\frac{d}{dx} h(x) = \frac{d}{dx} \int_0^{x^2} \sin z \, dz$$

not x let $x^2 = u$

$$= \frac{d}{dx} \int_0^{u} \sin z \, dz$$

$$= \sin u \cdot \frac{du}{dx}$$

Chain Rule!

$$h'(x) = \sin x^2 \,(2x)$$

$$h'(2) = 4 \sin 4 = -3.027\ldots$$

Example: Determine a function $y(x)$ such that

$$\frac{dy}{dx} = \sqrt{\tan x}$$

$$\frac{d}{dx}\left[\boxed{}\right] = \sqrt{\tan x}$$

$y(x)$ hard...

$$\int_a^x \sqrt{\tan t} \, dt$$

So easy to write :)

Note: Finding such a function without an integral is possible and is in fact:

$$y = \frac{1}{\sqrt{2}}\left[\arctan\left(\sqrt{2\tan x} - 1\right) + \arctan\left(\sqrt{2\tan x} + 1\right) + \frac{1}{2}\ln\left(\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right)\right]$$

But arguably, the simple integral does a good job and we can graph it and do computations with it thanks to computers.

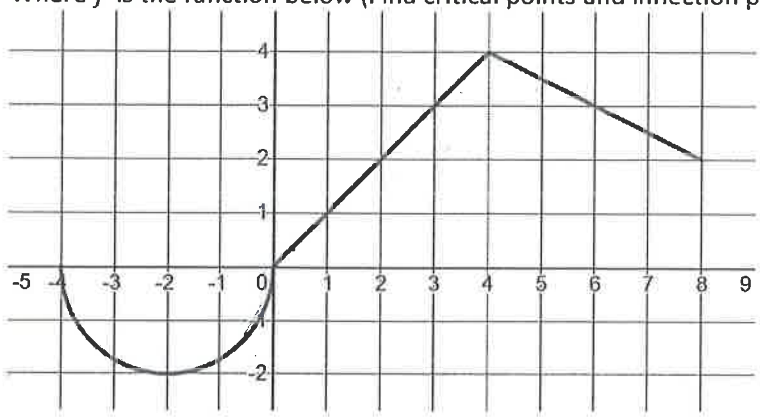
Practice: Accurately sketch the curve

$$F(x) = \int_{-4}^{x} f(u)du$$

Where f is the function below (Find critical points and inflection points)

$F'(x) = 0$?

$F''(x) = 0$? / undefined



$F'(x) = f(x) \rightarrow F''(x) = f'(x)$

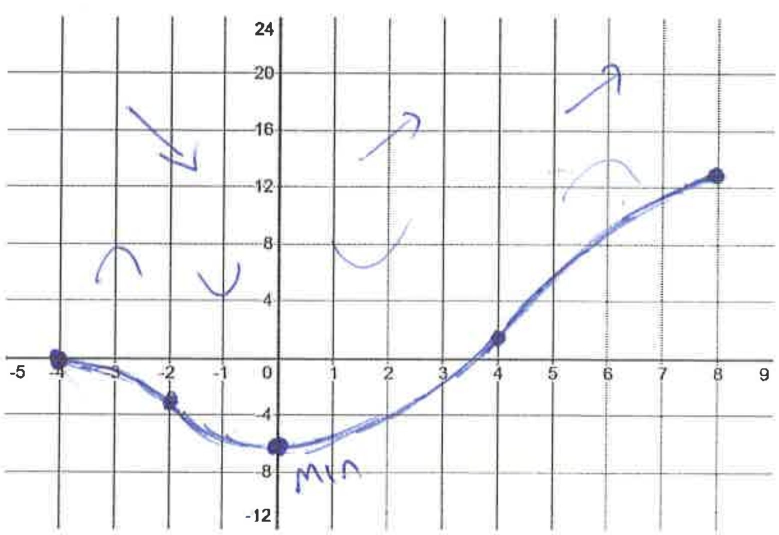
$f(x) = 0$ @ $x = -4, 0$

max / min

$f'(x) = 0$ / undefined

@ $x = -2, 0, 4$

change sign



2
1
4
3

Practice Problems: 5.4 # 37-46, 48-50, 53-56, 60

64

Look Ahead: How can the Fundamental Theorem be used to evaluate $\int_a^b f(x)dx$?