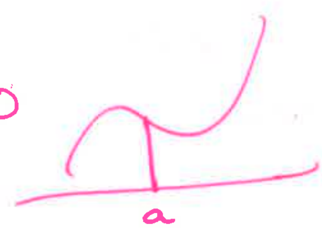
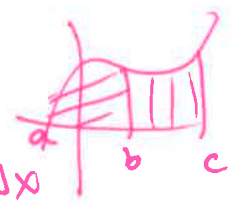
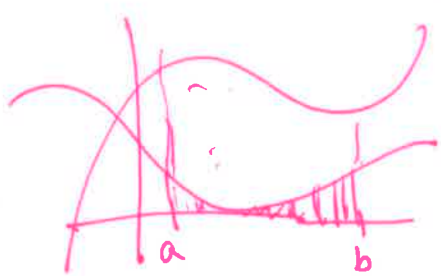
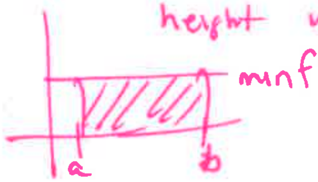
Integral Properties and Mean Value Theorem

Goal: • Can use limit properties with Riemann sums and can justify integral properties using area. • Can find the average value of a function
Terminology: • Mean Value Theorem for Integrals

Integral Properties

Property	Proof or Justification
$\int_a^b f(x)dx = -\int_b^a f(x)dx$ Backwards is negative	on $\int_b^a f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$ where $\Delta x_k = x_k - x_{k-1}$ and $P = \{a = x_0, x_1, \dots, x_n = b\}$ so $\Delta x_k < 0$ but $I = -\lim_{\|P\|\to 0} \sum f(c_k)\delta x_k$ where $\delta x_k = -\Delta x_k$ $I = -\int_a^b f(x)dx$
$\int_a^a f(x)dx = 0$ 0 width	$P = \{a = x_0, \dots, x_n = a\}$ Then $P = \{a\}$ so $\Delta x = 0$ $\lim_{\|P\|\to 0} \sum f(c_k)\Delta x_k = 0$ 
$\int_a^b kf(x)dx = k\int_a^b f(x)dx$ Constants factor out	$\lim_{\|P\|\to 0} \sum_n k f(c_n)\Delta x_n = \lim_{\|P\|\to 0} k \sum_n f(c_n)\Delta x_n$ $= k \lim_{\|P\|\to 0} \sum_n f(c_n)\Delta x_n$ $= k\int_a^b f(x)dx$

$\int_a^b (f(x) \pm g(x))dx$ $= \int_a^b f(x)dx \pm \int_a^b f(x)dx$ same total height	$\lim_{\|P\|\to 0} \sum (f(c_k)+g(c_k))\Delta x_k$ $= \lim_{\|P\|\to 0} \left[\sum f(c_k)\Delta x_k + \sum g(c_k)\Delta x_k\right]$ $= \underbrace{\lim_{\|P\|\to 0} \sum f(c_k)\Delta x_k}_{\text{as long as this exists}} + \lim_{\|P\|\to 0} \sum g(c_k)\Delta x_k$ $= \int_a^b f(x)dx + \int_a^b g(x)dx$
$\int_a^c f(x)dx$ $= \int_a^b f(x)dx + \int_b^c f(x)dx$ same total width	$P = \{a = x_0, \ldots, b = x_k, \ldots, c = x_n\}$ $Q = \{a = x_0, \ldots, x_k\}$ $R = \{x_k, \ldots, x_n\}$ $\lim_{\|P\|\to 0} = \lim_{\substack{\|Q\|\to 0 \\ \|R\|\to 0}}$  $\lim_{\|P\|\to 0} \sum f(x)\Delta x = \lim_{\substack{\|Q\|\to 0 \\ \|R\|\to 0}} \overbrace{\sum f(x)\Delta x}^{\text{sum}}$ $= \lim_{\|Q\|\to 0} \sum f(x)\Delta x + \lim_{\|R\|\to 0} \sum f(x)dx$
If $f(x) \le g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \le \int_a^b g(x)dx$ Boundedness	$\lim_{\|P\|\to 0} \sum f(x)\Delta x \le \lim_{\|P\|\to 0} \sum g(x)\Delta x$ since $f(x) \le g(x)$ 

$\min f \cdot (b-a) \le \int_a^b f(x)dx$ $\int_a^b f(x)dx \le \max f \cdot (b-a)$	$\min f \le f(x) \le \max f$ so $\int_a^b \underbrace{\min f}_{\text{const}}\, dx \le \int_a^b f(x)\,dx$ $\underbrace{\min f}_{\text{height}}\ \underbrace{(b-a)}_{\text{width}} \le \int_a^b f(x)\,dx$ 
$avg(f) = \frac{1}{b-a}\int_a^b f(x)dx$	$avg\ f = \lim_{\|P\|\to 0} \sum_{n=1}^{n} \frac{f(c_k)}{n} \cdot \frac{\Delta x_k}{\Delta x_k}$ make $\Delta x_k = \frac{b-a}{n}$ $\forall$ k $\Rightarrow avg\ f = \lim_{\|P\|\to 0} \frac{1}{\cancel{n}\left(\frac{b-a}{\cancel{n}}\right)} \sum f(c_k)\Delta x$ $= \frac{1}{b-a}\int_a^b f(x)dx$ height width Area ~~height~~

The **Mean Value Theorem** for integrals states the following:
If f is continuous on $[a,b]$, then there exists some point $c \in [a,b]$ such that

$$f(c) = avg(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Proof: f is continuous on $[a,b]$ and ~~reaches~~

~~$f(x) \in [f(a), f(b)]$~~ $f(x) \in [\min f, \max f]$

clearly $avg\ f \in [\min f, \max f]$ so by IVT

$\exists\ c \in [a,b]$ s.t $f(c) = avg\ f$

Practice Problems: 5.3 # 1-6, 25-28 (use fnInt), 29-31, 33, 34, 36

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Look Ahead: How is the area related to a rate of change?