Series

1. If the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n, which of the following must be true?

(A) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$

(B) $|a_n| < 1$ for all n

(C) $\sum_{n=1}^{\infty} a_n = 0$

(D) $\sum_{n=1}^{\infty} na_n$ diverges

(E) $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges ✓

2. Which of the following series diverge?

I. $\sum_{n=0}^{\infty} \left(\frac{\sin 2}{\pi}\right)^n$

II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

III. $\sum_{n=1}^{\infty} \left(\frac{e^n}{e^n + 1}\right)$

(A) III only

(B) I and II only

(C) I and III only

(D) II and III only ✓

(E) I, II, and III

Series

3. If $f(x) = \sum_{k=1}^{\infty} \left(\sin^2 x\right)^k$, then $f(1)$ is

(A) 0.369

(B) 0.585

(C) 2.400

(D) 2.426 ✓

(E) 3.426

4. What is the value of $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{5^n}$?

(A) $-\frac{15}{8}$

(B) $-\frac{9}{8}$

(C) $-\frac{3}{8}$

(D) $\frac{9}{8}$ ✓

(E) $\frac{15}{8}$

Series

5. The nth term test can be used to determine divergence for which of the following series?

1. $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$
2. $\sum_{k=0}^{\infty} (-1)^k \left(\frac{k}{2k+1}\right)$
3. $\sum_{k=1}^{\infty} \frac{3k^2 - k^3}{5k^3}$

(A) III only

(B) I and III only

(C) II and III only ✓

(D) I, II, and III

6. Let f be a positive, continuous, decreasing function. If $\int_1^{\infty} f(x)\, dx = 5$, which of the following statements about the series $\sum_{n=1}^{\infty} f(n)$ must be true?

(A) $\sum_{n=1}^{\infty} f(n) = 0$

(B) $\sum_{n=1}^{\infty} f(n)$ converges, and $\sum_{n=1}^{\infty} f(n) < 5$

(C) $\sum_{n=1}^{\infty} f(n) = 5$

(D) $\sum_{n=1}^{\infty} f(n)$ converges, and $\sum_{n=1}^{\infty} f(n) > 5$ ✓

(E) $\sum_{n=1}^{\infty} f(n)$ diverges

Series

7. The integral test can be used to determine that which of the following statements about the infinite series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ is true?

- (A) The series converges because $\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2} \square x = -1 + e$. ✓
- (B) The series converges because $\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2} \square x = e$.
- (C) The series converges because $\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2} \square x = 1 - e$.
- (D) The series diverges because $\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2} \square x$ is not finite.

Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$, where $p \geq 0$.

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8. Determine whether the series converges or diverges for $p=1$. Show your analysis.

Please respond on separate paper, following directions from your teacher.

Part B

1 point is earned for anti derivative

1 point is earned for integral diverges

2 point is earned for conclusion with monotonically decreasing to 0.

$$Let\ f(x) = \frac{1}{x \ln x},\ so\ series\ is\ \sum_{2}^{\infty} f(n)$$

$$\int_{2}^{\infty} \frac{1}{x \ln x} = \lim_{b \to \infty} \ln |\ln x| \,\Big|_2^b = \lim_{b \to \infty} \ln \ln (b) - \ln \ln 2 = \infty$$

Since f(x) monotonically decreases to 0, the integral test shows $\sum_{2}^{\infty} \frac{1}{n \ln n}$ diverges.

Series

0	1	2	3	4

The student response earns four of the following points:

1 point is earned for anti derivative

1 point is earned for integral diverges

2 point is earned for conclusion with monotonically decreasing to 0.

Let $f(x) = \frac{1}{x \ln x}$, *so series is* $\sum_{2}^{\infty} f(n)$

$$\int_{2}^{\infty} \frac{1}{x \ln x} = \lim_{b \to \infty} \ln|\ln x| \,\Big|_2^b = \lim_{b \to \infty} \ln \ln(b) - \ln \ln 2 = \infty$$

Since f(x) monotonically decreases to 0, the integral test shows $\sum_{2}^{\infty} \frac{1}{n \ln n}$ diverges.

9. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

II. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

III. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$

Series

(A) None

(B) II only

(C) III only

(D) I and II only

(E) II and III only ✓

10. What are all values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}+n}$ diverges?

(A) $p \leq 1/2$ ✓

(B) p < 1/2 only

(C) $p \geq 1/2$

(D) p > 1/2 only

(E) The series diverges for all p.

11. For what values of p will both series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ and $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ converges?

(A) $-2 < p < 2$ only

(B) $-\frac{1}{2} < p < \frac{1}{2}$ only

(C) $\frac{1}{2} < p < 2$ only ✓

(D) $p < \frac{1}{2}$ and $p > 2$

(E) There are no such values of p.

12. What are all values of p for which $\int_1^\infty \frac{1}{x^{2p}}\,dx$ converges?

(A) $p < -1$

(B) $p > 0$

(C) $p > \frac{1}{2}$ ✓

(D) $p > 1$

(E) There are no values of p for which this integral converges.

13. Which of the following is a convergent p-series?

Series

(A) $\sum_{n=1}^{\infty} n^3$

(B) $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

(C) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(D) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^3$ ✓

14. Which of the following series converge?

$I. \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad II. \sum_{n=1}^{\infty} \frac{1}{n}$ $\quad III. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(A) I only

(B) III only

(C) I and II only

(D) I and III only ✓

(E) I, II, and III

15. Which of the following series diverge?

I. $\sum_{k=3}^{\infty} \frac{2}{k^2+1}$

II. $\sum_{k=1}^{\infty} \left(\frac{6}{7}\right)^k$

III. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k}$

Series

(A) None ✓

(B) II only

(C) III only

(D) I and III

(E) II and III

16. Which of the following series converges?

(A) $\sum_{n=1}^{\infty} \frac{3n}{n+2}$

(B) $\sum_{n=1}^{\infty} \frac{3n}{n^2+2}$

(C) $\sum_{n=1}^{\infty} \frac{3n}{n^2+2n}$

(D) $\sum_{n=1}^{\infty} \frac{3n^2}{n^3+2n}$

(E) $\sum_{n=1}^{\infty} \frac{3n^2}{n^4+2n}$ ✓

17. Which of the following series can be used with the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n - n^2}$ converges or diverges?

Series

(A) $\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\sum_{n=1}^{\infty} \frac{1}{4^n}$

(C) $\sum_{n=1}^{\infty} \frac{1}{5^n}$

(D) $\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ ✓

18. Which of the following series can be used with the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ converges or diverges?

(A) $\sum_{n=1}^{\infty} \frac{1}{n}$ ✓

(B) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(C) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

(D) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

19. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} a_n$ converges, which of the following must be true?

(A) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(B) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

(C) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ converges. ✓

(D) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

(E) If $b_n \leq a_n$, then the behavior of $\sum_{n=1}^{\infty} b_n$ cannot be determined from the information given.

20. If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$ for all n, which of the following statements must be true?

(A) $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(B) $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

(C) $\sum_{n=1}^{\infty} (-1)^n b_n$ diverges.

(D) $\sum_{n=1}^{\infty} b_n$ converges.

(E) $\sum_{n=1}^{\infty} b_n$ diverges. ✓

Let $a_n = \frac{1}{n \ln n}$ for $n \geq 3$.

---

21. Consider the infinite series $\sum_{n=3}^{\infty} (-1)^{n+1} a_n = \frac{1}{3 \ln 3} - \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} - \cdots$. Identify the properties of this series that guarantee the series coverage. Explain why the sum of this series is less than $\frac{1}{3}$.

Series

Please respond on separate paper, following directions from your teacher.

Part B

The response can earn up to 2 points:
1 point: properties
1 point: explanation

The terms in this alternating series decrease in absolute value and lim n→∞ 1 nlnn =0 . Therefore, the Alternating Series Test guarantees that this series converges. Furthermore,

$$\frac{1}{3In\ 3} - \frac{1}{4In\ 4} < \text{Sum} < \frac{1}{3In\ 3} < \frac{1}{3}$$

Therefore, the sum of the series is less than 1 3 .

0	1	2 ✓

The response can earn up to 2 points:
1 point: properties
1 point: explanation

The terms in this alternating series decrease in absolute value and lim n→∞ 1 nlnn =0 . Therefore, the Alternating Series Test guarantees that this series converges. Furthermore,

$$\frac{1}{3In\ 3} - \frac{1}{4In\ 4} < \text{Sum} < \frac{1}{3In\ 3} < \frac{1}{3}$$

Therefore, the sum of the series is less than 1 3 .

22. The power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n n^2}$ has radius of convergence 2. At which of the following values of *x* can the alternating series test be used with this series to verify convergence at *x* ?

Series

(A) 6

(B) 4 ✓

(C) 2

(D) 0

(E) -1

23. Which of the following statements are true about the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \frac{(-1)^n}{\sqrt{n}+(-1)^n}$?

I. The series is alternating.
II. $|a_{n+1}| \leq |a_n|$ for all $n \geq 2$
III. $\lim_{n\to\infty} a_n = 0$

(A) None

(B) I only

(C) I and II only

(D) I and III only ✓

(E) I, II, and III

Series

24. The alternating series test can be used to show convergence for which of the following series?

1. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots+a_n+\cdots$, where $a_n=(-1)^{n+1}\frac{1}{n^2}$

2. $\sin 1-\frac{\sin 2}{4}+\frac{\sin 3}{9}-\frac{\sin 4}{16}+\frac{\sin 5}{25}-\frac{\sin 6}{36}+\cdots+b_n+\cdots$, where $b_n=(-1)^{n+1}\frac{\sin n}{n^2}$

3. $\frac{1}{\sqrt{2}+1}-\frac{1}{\sqrt{2}-1}+\frac{1}{\sqrt{3}+1}-\frac{1}{\sqrt{3}-1}+\frac{1}{\sqrt{4}+1}-\frac{1}{\sqrt{4}-1}+\cdots+c_n+\cdots$,

where $c_n=\begin{cases}\frac{1}{\sqrt{k+1}+1} & \text{if } n=2k-1\\ -\frac{1}{\sqrt{k+1}-1} & \text{if } n=2k\end{cases}$

(A) I only ✓

(B) II only

(C) I and II only

(D) I and III only

25. Which of the following series converges for all real numbers *x* ?

(A) $\sum_{n=1}^{\infty}\frac{x^n}{n}$

(B) $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$

(C) $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$

(D) $\sum_{n=1}^{\infty}\frac{e^n x^n}{n!}$ ✓

(E) $\sum_{n=1}^{\infty}\frac{n!x^n}{e^n}$

Series

26. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$

II. $\sum_{n=1}^{\infty} \frac{n!}{n^{100}}$

III. $\sum_{n=1}^{\infty} \frac{n+1}{(n)(n+2)(n+3)}$

(A) I only

(B) II only

(C) III only

(D) I and III only ✓

(E) I, II, and III

27. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{n3^n}{x^n}$ converges?

(A) All x except $x = 0$

(B) $|x| = 3$

(C) $-3 \leq x \leq 3$

(D) $|x| > 3$ ✓

(E) The series diverges for all x.

Series

28. What are all positive values of p for which the series $\sum_{n=1}^{\infty} \frac{n^p}{4^n}$ will converge?

(A) $p > 0$ ✓

(B) $0 < p < 4$ only

(C) $p > 1$ only

(D) There are no positive values of p for which the series will converge.

29. Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. If the ratio test is applied to the series, which of the following inequalities results, implying that the series converges?

(A) $\lim_{n\to\infty} \frac{e}{n!} < 1$

(B) $\lim_{n\to\infty} \frac{n!}{e} < 1$

(C) $\lim_{n\to\infty} \frac{n+1}{e} < 1$

(D) $\lim_{n\to\infty} \frac{e}{n+1} < 1$ ✓

(E) $\lim_{n\to\infty} \frac{e}{(n+1)!} < 1$

Series

30. Which of the following series are conditionally convergent?

i. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

ii. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

iii. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(A) I only

(B) I and II only

(C) I and III only ✓

(D) II and III only

31. For what values of p is the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^p + 2}$ conditionally convergent?

(A) $0 < p \leq 1$

(B) $p > 1$

(C) $1 < p \leq 2$ only ✓

(D) $p > 2$ only

Let *f* be the function given by $f(x) = e^{-2x^2}$.

32. Let *g* be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x=0$. Show that $|f(x)-g(x)| < 0.02$ for $-0.6 \leq x \leq 0.6$.

Series

Please respond on separate paper, following directions from your teacher.

Part C

1 point is earned for correctly alternating series bound of $\frac{16x^8}{4!}$

$$f(x) - g(x) = \frac{16x^8}{4!} - \frac{32x^{16}}{5!} + \cdots$$

1 point is earned for correctly using $x = 0.6$

This is an alternating series for each x, since powers of x are even.

Also, $\left|\frac{a_n+1}{a_n}\right| = \frac{2}{n+1}x^2 < 1 \text{ for } -0.6 \leq x \leq 0.6$ so terms are decreasing in absolute value

1 point is used for the correct conclusion

Thus $\left|f(x) - g(x)\right| \leq \frac{16x^8}{4!} \leq \frac{16(0.6)^8}{4!}$
$= 0.011 \cdots < 0.02$

0	1	2	3 ✓

The student response earns three of the following points:

1 point is earned for correctly alternating series bound of $\frac{16x^8}{4!}$

$$f(x) - g(x) = \frac{16x^8}{4!} - \frac{32x^{16}}{5!} + \cdots$$

1 point is earned for correctly using $x = 0.6$

This is an alternating series for each x, since powers of x are even.

Also, $\left|\frac{a_n+1}{a_n}\right| = \frac{2}{n+1}x^2 < 1 \text{ for } -0.6 \leq x \leq 0.6$ so terms are decreasing in absolute value

1 point is used for the correct conclusion

Thus $\left|f(x)-g(x)\right| \leq \frac{16x^8}{4!} \leq \frac{16(0.6)^8}{4!}$
$= 0.011\cdots < 0.02$

33. The Taylor series for a function *f* about $x = 0$ is given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}x^{2n}$ and converges to *f* for all real numbers *x*. If the fourth-degree Taylor polynomial for *f* about $x = 0$ is used to approximate $f\left(\frac{1}{2}\right)$ alternating series error bound?

(A) $\frac{1}{2^4 \cdot 5!}$

(B) $\frac{1}{2^5 \cdot 6!}$

(C) $\frac{1}{2^6 \cdot 7!}$ ✓

(D) $\frac{1}{2^{10} \cdot 11!}$

The function *f* is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers *x*.

---

34. Show that $1 - \frac{1}{3!}$ approximates *f*(1) with error less than $\frac{1}{100}$.

Please respond on separate paper, following directions from your teacher.

Part B

1 point is earned for correctly showing error bound < $\frac{1}{100}$ $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots + \frac{(-1)^n}{(2n+1)!} + \ldots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left|f(1)-\left(1-\frac{1}{3!}\right)\right| \leq \frac{1}{5!}=\frac{1}{120}<\frac{1}{100}$.

0	1

The student response earns one of the following points:

1 point is earned for correctly showing error bound $< \frac{1}{100}$

$f(1)=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\ldots+\frac{(-1)^n}{(2n+1)!}+\ldots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left|f(1)-\left(1-\frac{1}{3!}\right)\right| \leq \frac{1}{5!}=\frac{1}{120}<\frac{1}{100}$.

35. If the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$ is approximated by the partial sum with 15 terms, what is the alternating series error bound?

(A) $\frac{1}{15}$

(B) $\frac{1}{16}$

(C) $\frac{1}{31}$

(D) $\frac{1}{33}$ ✓