Series

The function f satisfies the equation $f'(x) = f(x) + x + 1$ and $f(0) = 2$. The Taylor series for f about $x = 0$ converges to $f(x)$ for all x.

1. Find $f''(0)$ and find the second-degree Taylor polynomial for f about $x = 0$.

Please respond on separate paper, following directions from your teacher.

Part B

One point is earned for $f''(0)$

One point is earned for the second-degree Taylor polynomial

$$f''(x) = f'(x) + 1; f''(0) = f'(0) + 1 = 3 + 1 = 4$$
$$P_2(x) = 2 + 3x + \frac{4}{2!}x^2 = 2 + 3x + 2x^2$$

0	1	2 ✓

The student earns all of the following points:

One point is earned for $f''(0)$

One point is earned for the second-degree Taylor polynomial

$$f''(x) = f'(x) + 1; f''(0) = f'(0) + 1 = 3 + 1 = 4$$
$$P_2(x) = 2 + 3x + \frac{4}{2!}x^2 = 2 + 3x + 2x^2$$

2. Find the fourth-degree Taylor polynomial for f about $x = 0$.

Please respond on separate paper, following directions from your teacher.

Part C

One point is earned for $f'''(0)$ and $f^{(4)}(0)$

One point is earned for the fourth-degree Taylor polynomial

Series

$f'''(x) = f''(x); f'''(0) = f''(0) = 4$
$f^{(4)}(x) = f'''(x); f^{(4)}(0) = f'''(0) = 4$
$P_4(x) = 2 + 3x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4$
$= 2+3x+2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4$

0	1	2 ✓

The student earns all of the following points:

One point is earned for $f'''(0)$ and $f^{(4)}(0)$

One point is earned for the fourth-degree Taylor polynomial

$f'''(x) = f''(x); f'''(0) = f''(0) = 4$
$f^{(4)}(x) = f'''(x); f^{(4)}(0) = f'''(0) = 4$
$P_4(x) = 2 + 3x + \frac{4}{2!}x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4$
$= 2+3x+2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4$

3. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Series

x	0	$0 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$	4	$4 < x < 6$	6
$g(x)$	1	Positive	4	Positive	6	Positive	8	Positive	4
$g'(x)$	3	Positive	0	Positive	4	Positive	0	Negative	-4
$g''(x)$	-4	Negative	0	Positive	0	Negative	-2	Negative	-2

Let g be a twice-differentiable function. The function g and its derivatives have the properties indicated in the table above.

(a) Find the average rate of change of g' over the interval $[0, 6]$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

(b) Find the x-coordinate of each point of inflection of the graph of g on the open interval $0 < x < 6$. Justify your answers.

Please respond on separate paper, following directions from your teacher.

(c) Use a right Riemann sum with the three subintervals $[0, 2]$, $[2, 3]$, and $[3, 4]$ to approximate $\int_0^4 g(x)\,dx$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

(d) Is the approximation from part (c) an overestimate or underestimate for $\int_0^4 g(x)\,dx$? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(e) Find the value of $\int_0^3 x \cdot g''(x)\,dx$. Show the computations that lead to your answer.

Series

Please respond on separate paper, following directions from your teacher.

(f) Write the second-degree Taylor polynomial for g about $x = 4$.

Please respond on separate paper, following directions from your teacher.

Part A

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

A correct numerical answer without supporting work does not earn this point.

The point is earned for a correct numerical difference in a correct quotient. The difference in the numerator must be shown either as $g'(6) - g'(0)$ or as $-4 - 3$.

The values of $g'(6)$ and $g'(0)$ must be pulled from the table; $\frac{g'(6) - g'(0)}{6}$ does not yet earn the point.

The answer does not need to be simplified but if simplified, it must be correct to earn the point.

0	1 ✓

The student response accurately includes the criteria below.

☐ Answer

Solution:

Average rate of change $= \frac{g'(6)-g'(0)}{6-0} = \frac{-4-3}{6-0} = -\frac{7}{6}$

Part B

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

First point:

Series

The first point is earned for stating that *g* has points of inflection at both $x = 2$ and $x = 3.$ No reasoning is needed to earn this point. Note that:

· Endpoints or points outside the interval $0 < x < 6$ do not affect the scoring. Reponses are not penalized for inclusion and discussion of points outside of the interval.

· If any other internal points are identified as points of inflection or if only one point of inflection is given, the first point is not earned.

· If $g\left(2\right)$ and/or $g\left(3\right)$ are given as the points of inflection, the response does not earn the first point but is eligible for the second point.

· If ordered pairs are presented, they must be correct to earn the first point. That is, the ordered pairs must be $\left(2,\ g\left(2\right)\right)$ and $\left(3,\ g\left(3\right)\right)$, or equivalently $\left(2,\ 4\right)$ and $\left(3,\ 6\right)$.

· Set notation may be used: $\{2,\ 3\}$ should be read as "the set containing $x = 2$ and $x = 3.$ " Using any other notation such as $[2,\ 3]$ does not earn the first point.

· Just writing $2,\ 3$ does not earn the point. The response must indicate that these are *x*-values.

Second point:

The second point is earned for the justification of both $x = 3$ and $x = 4.$ The justification must state that $g''\left(x\right)$ changes sign at these places to earn the point.

The justification does not have to specify the direction of the sign change at either point. However, if the response reports the wrong direction of the sign change in $g''\left(x\right)$ at either point, this point is not earned.

Special case: If a response presents one of the two correct points of inflection (and presents no other points) with a correct justification, then the response will earn one of the two points.

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ Answer
- ☐ Justification

Solution:

$g''\left(x\right)$ changes sign at $x = 2$ and $x = 3.$ Therefore, the graph of *g* has points of inflection at $x = 2$ and $x = 3.$

Series

Part C

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

First point:

The first point is earned for the setup of a right Riemann sum. The minimum requirement is that we must see the sum of a product of a height and a width for each of the three rectangles. The width of 1 can be implied and does not need to be shown.

The first point is earned if there is at most one error in the six factors of the terms of the sum $g(2)(2-0)+g(3)(3-2)+g(4)(4-3)$.

Responses that start with numerical values are eligible to earn the first point. For example, the following responses earn the first point:

- $4 \cdot 2+6 \cdot 2+8 \cdot 1$
- $4 \cdot 2+6+8$
- $a \cdot 2+6+8$, where $a \neq 4$
- $4 \cdot b+6+8$, where $b \neq 2$

In contrast, a response that only shows $8+6+8$ or 22 does not earn the first point.

Second point:

Only responses that earn the first point are eligible to earn the second point.

The second point is earned only for the correct answer (22). The answer does not need to be simplified, but if simplified, it must be correct to earn the second point.

Equating the actual value of the integral with the approximation will not result in a loss of a point.

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ Right Riemann sum
- ☐ Answer

Series

Solution:

$$\int_0^4 g(x)\ dx \approx g(2)(2-0) + g(3)(3-2) + g(4)(4-3)$$

$$= 4 \cdot 2 + 6 \cdot 1 + 8 \cdot 1 = 22$$

Part D

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

Note: Responses in part (d) must include a conclusion with reasoning to be eligible for points. A correct conclusion without reasoning earns no points.

First point:

The first point is earned for reasoning that *g* is increasing over the given interval.

To earn the first point the response must state $g'(x) > 0$ or $g'(x) \geq 0$. For clarification:

· Stating *g* is increasing, without stating $g'(x) > 0$ or $g'(x) \geq 0$, does not earn the first point.

· Stating $g'(x) > 0$ or $g'(x) \geq 0$, without stating that *g* is increasing, earns the first point.

A picture, graph, or sign chart must be accompanied by a verbal explanation that $g'(x) > 0$ or $g'(x) \geq 0$ in order to be relevant. Without a verbal explanation, a picture, graph, or sign chart does not earn either point.

Use of pronouns such as "it" or the "curve" are ambiguous and will not earn the point unless they are clearly tied to a correct use of a word such as derivative, slope of *g*, g', or *g*. Note that the word "function" will be read as meaning the function *g*.

Intervals are not required to earn the first point. If intervals are presented, they must be $0 < x < 2$ and $2 < x < 4$, or $0 < x < 4$, with or without endpoints included.

Second point:

The second point is earned for reasoning and a correct conclusion.

Eligibility: The response must state either that $g'(x) > 0$ or that *g* is increasing.

Special case: If a response includes $g'(x) > 0$, reports that *g* was decreasing, and concludes that the right Riemann sum approximation is an underestimate, the response earns the second point, but not the first point.

0	1	2 ✓

Series

The student response accurately includes **both** of the criteria below.

- ☐ *g* increasing
- ☐ Answer

Solution:

$g'(x) > 0$ for $0 < x < 2$ and $2 < x < 4$, so *g* is increasing for $0 \le x \le 4$.

Therefore, the right Riemann sum approximation for $\int_0^4 g(x)\ dx$ is an overestimate.

Part E

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

First point:

The first point is earned for using integration by parts and presenting an expression in *x*. Limits of integration are not considered for the first point. Responses with correct, incorrect, or no limits are eligible to earn the first point. A single sign error between the two terms will still earn this point, but will not earn the third (answer) point. Responses that would earn the first point include:

- $x\,g'(x)\,|_0^3 - \int_0^3 g'(x)\ dx$
- $x\,g'(x)\,|_0^3 + \int_0^3 g'(x)\ dx$
- $x\,g'(x) - \int g'(x)\ dx$
- $x\,g'(x) + \int g'(x)\ dx$
- $x\,g'(x) \pm g(x)$

Any mixture of indefinite and definite integrals in these expressions will also earn the first point.

Second point:

Responses are eligible for the second point even if they did not earn the first point. The second point is earned for correctly using the Fundamental Theorem of Calculus to conclude that $\int_0^3 g'(x)\ dx = g(3) - g(0)$ or

Series

$$\int_0^3 g'(x)\ dx = 6 - 1.$$

Note that it is not required to have the remaining terms from the integration by parts present or correct to earn this point.

A response that begins with a correct presentation of all terms resulting from evaluation of the integral by integration by parts will earn the integration by parts point and the Fundamental Theorem of Calculus point. Note that the term $0 \cdot g'(0)$ must be present.

Responses that begin with:

· $3 \cdot g'(3) - 0 \cdot g'(0) - (g(3) - g(0))$ earns the first two points.

· $3 \cdot g'(3) - 0 - (g(3) - g(0))$ earns the first two points.

· $3 \cdot g'(3) - (g(3) - g(0))$ does not earn the first point, but does earn the second point.

· $3 \cdot g'(3) - 0 \cdot g'(0) - (g(3) + g(0))$ earns the first point, but does not earn the second point.

Third point:

The third point is earned only for the correct answer (7). Simplifying the arithmetic is not needed but if presented, must be correct to earn the third point.

Responses that include a "$+C$" on a definite integral are eligible for the first and second points (integration by parts and FTC) but cannot earn the answer point.

0	1	2	3 ✓

The student response accurately includes **all three** of the criteria below.

- ☐ Integration by parts
- ☐ Fundamental Theorem of Calculus
- ☐ Answer

Solution:

$u = x \qquad dv = g''(x)\ dx$

$du = dx \quad v = g'(x)$

Series

$$\int_0^3 x \cdot g''(x)\ dx = x \cdot g'(x)\,|_0^3 - \int_0^3 g'(x)\ dx$$

$$= 3 \cdot g'(3) - 0 \cdot g'(0) - (g(3) - g(0))$$

$$= 3 \cdot 4 - 0 \cdot 3 - (6 - 1) = 12 - 0 - 5 = 7$$

Part F

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The point is earned for the correct answer. No work needs to be shown. Simplifications of the coefficients if presented, must be correct to earn the point.

A response with additional terms beyond the quadratic term or an ellipsis $(+\cdots)$ at the end of the expression will not earn this point.

Equating $g(x)$ with the polynomial is acceptable for this point.

Responses that present $8-(x-4)^2$ and then go on to expand the polynomial to $8-16+8x+x^2$ or $-8+8x+x^2$ will earn the point unless the expanded form is declared as the final answer by circling or boxing.

0	1 ✓

The student response accurately includes the criteria below.

☐ Answer

Solution:

$$g(4) + g'(4)(x-4) + \frac{g''(4)}{2!}(x-4)^2$$
$$= 8 + 0(x-4) + \frac{-2}{2}(x-4)^2 = 8-(x-4)^2$$

4. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

x	4	4.2	4.4
$g'(x)$	32	50	72

Let $y = g(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = g'(x)$ with initial condition $g(4) = 3$. Selected values of g' are given in the table above. The function g is three-times differentiable.

(a) Write an equation for the line tangent to the graph of g at $x = 4$.

Please respond on separate paper, following directions from your teacher.

(b) Let h be the function defined by $h(x) = 15 + x \cdot g(x)$. Find $h'(4)$.

Please respond on separate paper, following directions from your teacher.

(c) Use a right Riemann sum with the two subintervals indicated by the data in the table to approximate $\int_4^{4.4} g'(x)\,dx$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

(d) It is known that $g''(x) > 0$ for $4 \leq x \leq 4.4$. Is the approximation from part (c) an overestimate or an underestimate for $\int_{4}^{4.4} g'(x)\,dx$? Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(e) Use the approximation for $\int_{4}^{4.4} g'(x)\,dx$ from part (c) to estimate the value of $g(4.4)$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

(f) Use Euler's method, starting at $x = 4$ with two steps of equal size, to approximate $g(4.4)$. Show the computations that lead to your answer.

Please respond on separate paper, following directions from your teacher.

(g) It is known that $g''(4) = 80$ and $g'''(4) = 100$. Write the third-degree Taylor polynomial for g about $x = 4$.

Please respond on separate paper, following directions from your teacher.

Part A

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The response must use the numerical values 3 and 32.

As there is only one point for this part, any simplification errors will not earn the point, even in the presence of a correct answer.

We do not need to see the first line of the model solution. A correct tangent line will earn the point.

Even though we have defined the function $g(x)$ in the stem of the problem, we will accept responses that replace $g(x)$ in place of *y* in the tangent line equation. For example, $g(x) = 32(x - 4) + 3$ earns the point.

All correct forms of the tangent line earn this point. For example:

Series

· $y - 3 = 32\left(x - 4\right)$

· $y = 3 + 32\left(x - 4\right)$

· $y = 32x - 128 + 3$

· $y = 32x - 125$

0	1

The student response accurately includes the criteria below.

☐ Answer

Solution:

$y = g\left(4\right) + g'\left(4\right)\left(x - 4\right)$

$y = 3 + 32\left(x - 4\right)$

Part B

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

Once you see a correct application of the product rule, the response has earned the point. If the response continues and makes an error in simplifying, the response will not earn the second (answer) point.

The first point is for an essentially correct application of the product rule. The point is earned only for a derivative that is one of the following:

· $h'\left(x\right) = g\left(x\right) + x\,g'\left(x\right)$; or

· $h'\left(x\right) = 15 + g\left(x\right) + x\,g'\left(x\right)$.

Note that the mistake with the derivative of the constant in the second example above will make the response ineligible for the second point.

To earn the answer point, we must see a numerical value equivalent to 131. Any arithmetic or algebraic errors in simplification will not earn the answer point, even in the presence of the correct answer.

A response of $3 + 4 \cdot 32$ is the minimal work required to earn both points.

Responses of just $3 + 128$ or just 131 would not earn either point because there is no supporting work.

Responses with some values of x plugged in, but not all, may still earn both points. For example, $h'(4) = g(4) + x\,g'(4) = 131$ will earn both points.

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ Product rule
- ☐ Answer

Solution:

$h'(x) = 1 \cdot g(x) + x \cdot g'(x)$

$h'(4) = 1 \cdot g(4) + 4 \cdot g'(4) = 1 \cdot 3 + 4 \cdot 32 = 131$

Part C

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

A response must demonstrate the Riemann sum in order to earn this point. This means we must see a sum of two products.

· This could be done with a response of just $50 \cdot 0.2 + 72 \cdot 0.2$ or $0.2(50 + 72)$.

· A response of just $10 + 14.4$ does not earn the point.

The answer must be presented numerically. Responses that stop with $g'(4.2) \cdot 0.2 + g'(4.4) \cdot 0.2$ do not earn the point, as we need to see values pulled from the given table.

The answer does not need to be simplified to 24.4, but if the answer is simplified, this must be done correctly.

0	1 ✓

The student response accurately includes the criteria below.

☐ Right Riemann sum and answer

Solution:

$$\int_{4}^{4.4} g'(x)\ dx \approx g'(4.2)(4.2-4)+g'(4.4)(4.4-4.2)$$
$$= 50 \cdot 0.2 + 72 \cdot 0.2 = 24.4$$

Part D

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The first point requires a statement that is equivalent to "g' increasing," but that is not simply a reference to concavity. This will require describing the behavior of g' or g. For example, it will suffice to say, "the slopes of g are increasing," to earn the first point. However, saying that "g is concave up" will not suffice by itself.

A response of just $g'' > 0$ without a conclusion of how this is tied to g' does not earn the first point.

Ambiguous statements will not earn the first point. For example, it is not sufficient to say, "the slopes are increasing," as it is not clear which function is being referenced. References to "the graph" or "the curve" will also be viewed as ambiguous, as these terms have not been defined in the problem.

Ambiguous use of "the function" will be interpreted as the function g as this is the only "function" defined in the stem. If a response earns the first point and goes on to say something incorrect, the response will not earn the second point.

If the response earns the first point and goes on to say something irrelevant (but not incorrect), the response is still eligible to earn the second point.

Statements whose truth cannot be determined from the information given will also not earn the second point. There are a variety of these that you may see, such as "g'' is concave up" or "g' is concave up."

It is not possible to earn the second point without earning the first point, even in the presence of the correct conclusion of "overestimate."

Special case: If a response has used $g'' > 0$ to incorrectly conclude that g' is decreasing on the interval, and then answers that the approximation is an underestimate, the response will earn one point.

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ g' increasing
- ☐ Answer

Solution:

Because $g''(x) > 0$ for $4 \leq x \leq 4.4$, g' is increasing on this interval.

Therefore, the right Riemann sum approximation for $\int_4^{4.4} g'(x)\ dx$ is an overestimate.

Part E

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

This point is earned only for a numerical answer. This answer does not need to be simplified, but any attempts at simplification must be correct.

The response must show at least the computation $3 + 24.4$ to earn this point. An answer of just 27.4 would not earn the point.

The response will not be penalized for failing to use $\approx$.

Responses are expected to use the approximation of the definite integral found in part (c). A response may do this even with an incorrect approximation found in part (c). A response may import an incorrect answer from part (c) and still earn the point in part (e).

The only acceptable answer in part (e) is the correct answer of 27.4 or an answer that is consistent with the approximation value found in part (c). (Note that when using an incorrect approximation, the minimal work required would be $3 + (\textbf{answer from part (c)})$.)

0	1 ✓

The student response accurately includes the criteria below.

- ☐ Answer

Solution:

Series

$$\int_4^{4.4} g'(x)\ dx = g(4.4) - g(4)$$

$$g(4.4) = g(4) + \int_4^{4.4} g'(x)\ dx \approx 3 + 24.4 = 27.4$$

Part F

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The first point is awarded for a response that provides clear evidence of an understanding of Euler's method, presented in a table, or in the form shown in the model solution. This clear understanding would be demonstrated by no more than one error in the two steps in Euler's method.

Responses that use $g'(4.2)$ in the first step and $g'(4.4)$ in the second step (rather than correctly using the values $g'(4)$ and $g'(4.2)$, will earn the first point as long as there are no additional errors. These responses are ineligible for the second point.

The answer point is earned only by the correct answer.

The answer does not need to be simplified; $3 + 32 \cdot 0.2 + 50 \cdot 0.2$ would earn both points.

An answer of 19.4 with no supporting work will earn no points. Similarly, an answer of only $9.4 + 10$ is not sufficient to earn any points.

A tabular answer with or without labels will earn both points if all values are correct, even if the final answer is found only in the table (and not circled or boxed).

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ First step of Euler's method
- ☐ Answer

Solution:

$$g(4.2) \approx g(4) + g'(4)(4.2 - 4) = 3 + 32 \cdot 0.2 = 9.4$$

$$g(4.4) \approx g(4.2) + g'(4.2)(4.4 - 4.2) \approx 9.4 + 50 \cdot 0.2 = 19.4$$

Part G

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

The coefficients of the presented polynomial do not need to be simplified. (However, errors in simplification would mean the response does not earn the second point.) For example, if a response presents the second line of the model solution, the response has earned both points.

Only polynomials centered at $x = 4$ are eligible for points in part (g). Polynomials centered about any value other than 4 will earn no points in this part (as only one term could be correct).

Responses with at least two terms correctly centered about $x = 4$, with numerical coefficients will earn the first point. Once earned, this point cannot be lost.

Including terms of degree greater than 3 or adding "$+ \ldots$" to the presented Taylor polynomial will not earn the second point.

Responses with terms multiplied out will still be eligible for both points, unless the response boxes the expanded polynomial, indicating that the polynomial about $(x-4)$ is not correct. For example, $3 + 32(x-4) + \frac{80}{2}(x-4)^2 + \frac{100}{6}(x-4)^3 = 32x - 125 + 40(x-4)^2 + \frac{50}{3}(x-4)^3$ earns both points, unless the right-hand side of the equation is circled or boxed.

Some responses choose to name the polynomial or equate it with our function $g(x)$. We will accept any naming of this polynomial, including $g(x)$, with one caveat. If the response gives it a name with an argument, the argument may not be a number. For example, responses with names such as $g(x)$ or $T_3(x)$ or T are eligible for both points. However, responses that equate the polynomial with $g(4)$ or $T(4)$ are ineligible for the second point.

Special case: A response with the first line of the model solution, $g(4) + g'(4)(x-4) + \frac{g''(4)}{2!}(x-4)^2 + \frac{g'''(4)}{3!}(x-4)^3$, will earn one point. (Note: the response has presented a correct form of the Taylor polynomial for $g(x)$ centered at $x = 4$.) Once this point has been earned, it cannot be lost. Any errors substituting numerical values or simplifying make the response ineligible to earn the second point.

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ Two terms
- ☐ Remaining terms

Solution:

Series

$$g(4) + g'(4)(x-4) + \frac{g''(4)}{2!}(x-4)^2 + \frac{g'''(4)}{3!}(x-4)^3$$
$$= 3 + 32(x-4) + \frac{80}{2}(x-4)^2 + \frac{100}{6}(x-4)^3$$
$$= 3 + 32(x-4) + 40(x-4)^2 + \frac{50}{3}(x-4)^3$$

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

5. Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

Please respond on separate paper, following directions from your teacher.

Part D

One point is earned for the Taylor polynomial

One point is earned for the approximation

$$T_2(x) = 15 + 8(x-1) + \frac{20}{2!}(x-1)^2$$
$$= 15 + 8(x-1) + 10(x-1)^2$$
$$f(1.4) \approx 15 + 8(1.4-1) + 10(1.4-1)^2 = 19.8$$

0	1	2 ✓

The student earns all of the following points:

One point is earned for the Taylor polynomial

One point is earned for the approximation

Series

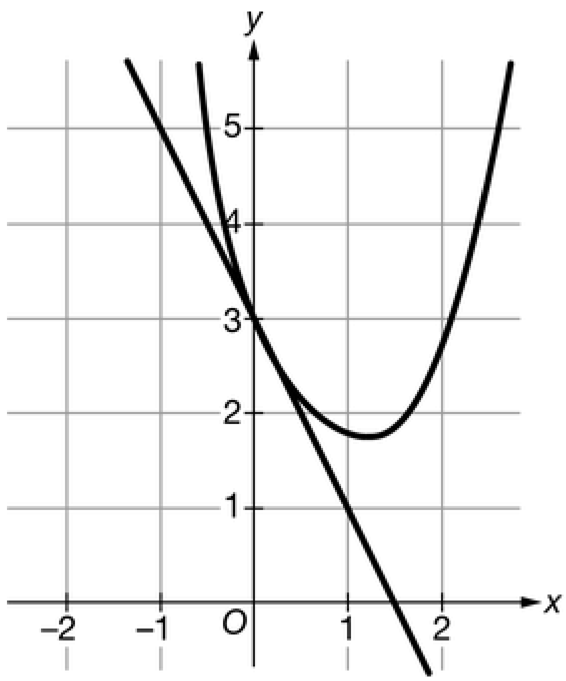
$T_2(x) = 15 + 8(x-1) + \frac{20}{2!}(x-1)^2$
$= 15 + 8(x-1) + 10(x-1)^2$
$f(1.4) \approx 15 + 8(1.4-1) + 10(1.4-1)^2 = 19.8$

6. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

Series

a) Write the third-degree Taylor polynomial for f about $x = 0$.

Please respond on separate paper, following directions from your teacher.

b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

Please respond on separate paper, following directions from your teacher.

c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

Please respond on separate paper, following directions from your teacher.

d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

Please respond on separate paper, following directions from your teacher.

Part A

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

The first point is earned for a response with two correct terms of the third-degree Taylor polynomial.

The second point is earned for having all four, and only those four, terms in the Taylor polynomial. Polynomials with more than four terms or that contain " $+\dots$ " do not earn the second point.

0	1	2 ✓

Series

The student response accurately includes **both** of the criteria below.

- ☐ two terms
- ☐ remaining terms

Solution:

$f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

Part B

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

The first point is earned for the terms $1 + x + \frac{1}{2!}x^2$. Listing the terms also earns the first point.

The second point is earned for the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$; a polynomial with degree higher than 2 does not earn the second point.

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ three terms for e^x
- ☐ three terms for $e^x f(x)$

Solution:

The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$3\left(1+x+\frac{1}{2!}x^2\right)-2x\,(1+x)+\frac{3}{2}x^2\,(1)$$
$$=3+(3-2)\,x+\left(\frac{3}{2}-2+\frac{3}{2}\right)x^2$$
$$=3+x+x^2.$$

Part C

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

A response imported from Part B that is third-degree and has at least two nonzero terms is eligible for the first point, as well as the second point, with correct, consistent work.

The first point is earned for the antiderivative $3-2t+\frac{3}{2}t^2-\frac{23}{12}t^3$. An antiderivative with a maximum of one error and four nonzero terms does not earn the first point, but is eligible to earn the second point with correct, consistent work.

Simplification is not required to earn the second point.

0	1	2 ✓

The student response accurately includes **both** of the criteria below.

- ☐ antiderivative
- ☐ answer

Solution:

$$h\,(1)=\int_0^1 f\,(t)\;dt$$
$$\approx\int_0^1\left(3-2t+\frac{3}{2}t^2-\frac{23}{12}t^3\right)\;dt$$
$$=\left[3t-t^2+\frac{1}{2}t^3-\frac{23}{48}t^4\right]_{t=0}^{t=1}$$
$$=3-1+\frac{1}{2}-\frac{23}{48}=\frac{97}{48}$$

Part D

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

Series

The first point is earned for $\int_0^x \left(\frac{54}{4!}t^4\right)\,dt$. Responses that show the result of antidifferentiation, such as $\frac{54}{5!}x^5$, also earn the first point. An error in antidifferentiation impacts the second point.

The second point is earned with use of the fifth-degree term of $h(1)$. Antidifferentiation must be correct to earn the second point. Simplification is not required to earn the second point.

The third point is earned by making an explicit link between the error, the calculated value, and the given value of 0.45.

0	1	2	3 ✓

The student response accurately includes **all three** of the criteria below.

- ☐ uses fourth-degree term of Maclaurin series for f
- ☐ uses first omitted term of series for $h(1)$
- ☐ error bound

Solution:

The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right)\,dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

Let $P(x)=7-3(x-4)+5(x-4)^2-2(x-4)^3+6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

---

7. Find $f(4)$ and $f'''(4)$.

Please respond on separate paper, following directions from your teacher.

Part A

1 point is earned for the correct answer $f(4) = 7$

$f(4) = P(4) = 7$

1 point is earned for the correct answer $f'''(4) = -12$

$\frac{f'''(4)}{3!} = -2, f'''(4) = -12$

0	1	2 ✓

The student response earns two of the following points:

1 point is earned for the correct answer $f(4) = 7$

$f(4) = P(4) = 7$

1 point is earned for the correct answer $f'''(4) = -12$

$\frac{f'''(4)}{3!} = -2, f'''(4) = -12$

8. Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

Please respond on separate paper, following directions from your teacher.

Part B

2 points are earned for the correct answer for $P'_3(x)$

< -1 > for each incorrect term

< -1 > for each additional term or + . . .

0/2 if degree < 2

$P_3(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3$

$P'_3(x) = -3 + 10(x-4) - 6(x-4)^2$

1 point is earned for the correct evaluation (only if degree 1, 2, 3)

Series

$f'(4.3) \approx -3+10(0.3)-6(0.3)\ 2 = -0.54$

0	1	2	3 ✓

The student response earns three of the following points:

2 points are earned for the correct answer for $P'_3(x)$

< -1 > for each incorrect term

< -1 > for each additional term or + . . .

0/2 if degree < 2

$P_3(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3$

$P'_3(x) = -3 + 10(x-4) - 6(x-4)^2$

1 point is earned for the correct evaluation (only if degree 1, 2, 3)

$f'(4.3) \approx -3+10(0.3)-6(0.3)\ 2 = -0.54$

Let f be the function given by $f(x) = e^{-2x^2}$.

9. Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

Please respond on separate paper, following directions from your teacher.

Part B

1 point is earned for the correct interval for e^x series

1 point is earned for making the correct connection

1 point is earned for the correct answer

Series

The series for e u converges for –∞<u<∞ So the series for e^{-2x^2} converges for $-\infty<-2x^2<\infty$ and thus, for –∞<x<∞

OR

1 point is earned for correctly setting up the ratio

1 point is earned for correctly computing the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{(-1)^n 2^n x^{2n}}\right|$$
$$= \lim_{n\to\infty}\frac{2}{n+1}x^2 = 0 < 1$$

1 point is earned for the correct answer

So the series for e^{-2x^2} converges for –∞<x<∞

0	1	2	3 ✓

The student response earns three of the following points:

1 point is earned for the correct interval for e^x series

1 point is earned for making the correct connection

1 point is earned for the correct answer

The series for e u converges for –∞<u<∞ So the series for e^{-2x^2} converges for $-\infty< -2x^2<\infty$ and thus, for –∞<x<∞

OR

1 point is earned for correctly setting up the ratio

1 point is earned for correctly computing the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{(-1)^n 2^n x^{2n}}\right|$$
$$= \lim_{n\to\infty}\frac{2}{n+1}x^2 = 0 < 1$$

1 point is earned for the correct answer

Series

So the series for e^{-2x^2} converges for $-\infty<x<\infty$

Let *f* be the function given by \(f\left(t\right)=\frac{4}{1+t^2}\) and *G* be the function given by \(G\left(x\right)=\int_0^xf\left(t\right)\ dt\).

10. Find the first four nonzero terms and the general term for the power series expansion of *G(x)* about *x*=0.

Please respond on separate paper, following directions from your teacher.

Part B

1 point is earned for first four terms of anti derivative

1 point is earned for general term of antiderivative

1 point is earned for evaluating antiderivative on [0,x] in power series context

$$G(x)=\int_0^x \frac{4}{1+t^2}dt=\int_0^x \left(4-4t^2+4t^4-4t^6+\cdots\right)\,dt$$

$$=\left(4t-\tfrac{4}{3}t^3+\tfrac{4}{5}t^5-\tfrac{4}{7}t^7+\cdots+\frac{(-1)^n 4t^{2n+1}}{2n+1}+\cdots\right)\Big|_0^x$$

$$=4x-\tfrac{4}{3}x^3+\tfrac{4}{5}x^5-\tfrac{4}{7}x^7+\cdots+\frac{(-1)^n 4x^{2n+1}}{2n+1}+\cdots$$

0	1	2	3 ✓

The student response earns three of the following points:

1 point is earned for first four terms of anti derivative

1 point is earned for general term of antiderivative

1 point is earned for evaluating antiderivative on [0,x] in power series context

Series

$$G(x) = \int_0^x \frac{4}{1+t^2}dt = \int_0^x \left(4 - 4t^2 + 4t^4 - 4t^6 + \cdots\right) dt$$

$$= \left(4t - \tfrac{4}{3}t^3 + \tfrac{4}{5}t^5 - \tfrac{4}{7}t^7 + \cdots + \frac{(-1)^n 4t^{2n+1}}{2n+1} + \cdots\right)\Big|_0^x$$

$$= 4x - \tfrac{4}{3}x^3 + \tfrac{4}{5}x^5 - \tfrac{4}{7}x^7 + \cdots + \frac{(-1)^n 4x^{2n+1}}{2n+1} + \cdots$$

11. Find the interval of convergence of the power series in part (b). (Your solution must include an analysis that justifies your answer.)

Please respond on separate paper, following directions from your teacher.

Part C

1 point is earned for setting up appropriate test

1 point is earned for correct conclusion (ratio test must include limit)

1 point is earned for tests both endpoints, invoking Alt. Series Test

Ratio Test,

$$\left|\frac{(-1)^{n+1}4X^{2n+3}}{2n+3} \cdot \frac{2n+1}{(-1)4X^{2n+1}}\right| = \frac{2n+1}{2n+3}X^2$$

$$\lim_{n\to\infty} \frac{2n+1}{2n+3}X^2 = X^2 \; ; \; X^2 < 1 \; for \; -1 < x < 1$$

$$G(1) = 4 - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \ldots Converges\ by\ Alternating\ Series\ Test$$

$$G(-1) = -4 + \tfrac{4}{3} - \tfrac{4}{5} + \ldots Converges\ by\ Alternating\ Series\ Test$$

Converges for $-1 \le x \le 1$

0	1	2	3 ✓

Series

The student response earns three of the following points:

1 point is earned for setting up appropriate test

1 point is earned for correct conclusion (ratio test must include limit)

1 point is earned for tests both endpoints, invoking Alt. Series Test

$$\left|\frac{(-1)^{n+1}4X^{2n+3}}{2n+3}\cdot\frac{2n+1}{(-1)4X^{2n+1}}\right| = \frac{2n+1}{2n+3}X^2$$

$$\lim_{n\to\infty}\frac{2n+1}{2n+3}X^2 = X^2\ ;\ X^2 < 1\ for\ -1 < x < 1$$

$$G(1) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots Converges\ by\ Alternating\ Series\ Test$$
$$G(-1) = -4 + \frac{4}{3} - \frac{4}{5} + \ldots Converges\ by\ Alternating\ Series\ Test$$

Converges for $-1 \leq x \leq 1$

12. What are all values of x for which the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n\cdot 3^n}$ converges?

(A) $-3 \leq x \leq 3$

(B) $-3<x<3$

(C) $-1 < x \leq 5$

(D) $-1 \leq x \leq 5$

(E) $-1 \leq x < 5$ ✓

The function g has derivatives of all orders, and the Maclaurin series for g is

Series

$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$

13. Using the ratio test, determine the interval of convergence of the Maclaurin series for *g*.

Please respond on separate paper, following directions from your teacher.

Part A

One point is earned for sets up ratio

One point is earned for computes limit of ratio

One point is earned for identifies interior of interval of convergence

One point is earned for considers both endpoints

One point is earned for analysis and interval of convergence

$$\left| \frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}} \right| = \left(\frac{2n+3}{2n+5} \right) \cdot x^2$$
$$\lim_{x \to \infty} \left(\frac{2n+3}{2n+5} \right) \cdot x^2 = x^2$$
$$x^2 < 1 \Rightarrow -1 < x < 1$$

The series converges when $-1 < x < 1$. When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$ This series converges by the Alternating Series Test. When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$ This series converges by the Alternating Series Test. Therefore, the interval of convergence is $-1 \le x \le 1$.

0	1	2	3	4	5 ✓

The student earns all of the following points:

One point is earned for sets up ratio

One point is earned for computes limit of ratio

Series

One point is earned for identifies interior of interval of convergence

One point is earned for considers both endpoints

One point is earned for analysis and interval of convergence

$$\left|\frac{x^{2n+3}}{2n+5}\cdot\frac{2n+3}{x^{2n+1}}\right|=\left(\frac{2n+3}{2n+5}\right)\cdot x^2$$
$$\lim_{x\to\infty}\left(\frac{2n+3}{2n+5}\right)\cdot x^2=x^2$$
$$x^2<1\Rightarrow -1<x<1$$

The series converges when $-1<x<1$. x When $x=-1$, the series is $-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$ This series converges by the Alternating Series Test. When $x = 1$, the series is $\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\cdots$ This series converges by the Alternating Series Test. Therefore, the interval of convergence is $-1\le x\le 1$.

14. What is the radius of convergence for the power series $\sum_{n=0}^{\infty}\frac{(x-4)^n}{2\cdot 3^{n+1}}$?

(A) $\frac{1}{3}$

(B) $\frac{3}{2}$

(C) 3 ✓

(D) 4

(E) 6

15. Let *f* be a function with second derivative f''(x) = $\sqrt{1+3x}$. The coefficient of x^3 in the Taylor series for *f* about *x* = *0* is

(A) $\frac{1}{12}$

(B) $\frac{1}{6}$

(C) $\frac{1}{4}$ ✓

(D) $\frac{1}{2}$

(E) $\frac{3}{2}$

16. What is the coefficient of x^6 in the Taylor series for $\frac{e^{3x^2}}{2}$ about $x = 0$?

(A) $\frac{1}{1440}$

(B) $\frac{81}{160}$

(C) $\frac{9}{4}$ ✓

(D) $\frac{9}{2}$

(E) $\frac{27}{2}$

17. If $f(x) = x \sin(2x)$, which of the following is the Taylor series for *f* aboutx=0 ?

Series

(A) $x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\cdots$

(B) $x-\frac{4x^3}{2!}+\frac{16x^5}{4!}-\frac{64x^7}{6!}+\cdots$

(C) $2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\cdots$

(D) $2x^2-\frac{2x^4}{3!}+\frac{2x^6}{5!}-\frac{2x^8}{7!}+\cdots$

(E) $2x^2-\frac{8x^4}{3!}+\frac{32x^6}{5!}-\frac{128x^8}{7!}+\cdots$ ✓

18. Which of the following is the Maclaurin series for $\frac{1}{(1-x)^2}$?

(A) $1-x+x^2-x^3+\ldots$

(B) $1-2x+3x^2-4x^3+\ldots$

(C) $1+2x+3x^2+4x^3+\ldots$ ✓

(D) $1+x^2+x^4+x^6+\ldots$

(E) $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots$

19. A function *f* has Maclaurin series given by $\frac{x^4}{2!}+\frac{x^5}{3!}+\frac{x^6}{4!}+\cdots+\frac{x^{n+3}}{(n+1)!}+\cdots$. Which of the following is an expression for $f(x)$?

(A) $-3x \sin x + 3x^2$

(B) $-\cos\left(x^2\right) + 1$

(C) $-x^2 \cos x + x^2$

(D) $x^2 e^x - x^3 - x^2$ ✓

(E) $e^{x^2} - x^2 - 1$

20. The sum of the series $1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots + \frac{2^n}{n!} + \ldots$ is

(A) ln 2

(B) e^2 ✓

(C) cos 2

(D) sin 2

(E) nonexistent

21. What is the sum of the series $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$?

Series

(A) $\ln 2$

(B) $\ln(1+\ln 2)$

(C) 2 ✓

(D) e^2

(E) The series diverges.

The function *f* is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!}+\cdots$$

for all real numbers *x*.

---

22. Show that $y=f(x)$ is a solution to the differential equation $xy'+y=\cos x$.

Please respond on separate paper, following directions from your teacher.

Part C

1 point is earned for the correct series for y' OR for the correct series for $xf(x)$

$$y'=-\frac{2x}{3!}+\frac{4x^3}{5!}-\frac{6x^5}{7!}+\ldots+\frac{(-1)^n 2nx^{2n-1}}{(2n+1)!}+\ldots$$

1 point is earned for the correct series for xy' OR for identifying the series as $\sin x$

$$xy'=-\frac{2x^2}{3!}+\frac{4x^4}{5!}-\frac{6x^6}{7!}+\ldots+\frac{(-1)^n 2nx^{2n}}{(2n+1)!}+\ldots$$

OR

Series

$\text{xy} = \text{xf}(x) = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \ldots$
$= \text{sinx}$

1 point is earned for the correct series for $xy' + y$ OR for handling $xy' + y$

$\text{xy'} + y = 1 - (\frac{2}{3!} + \frac{1}{3!})x^2 + (\frac{4}{5!} + \frac{1}{5!})x^4 - (\frac{6}{7!} + \frac{1}{7!})x^6 + \ldots$
$+ (-1)^n (\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!})x^{2n} + \ldots$

1 point is earned for identifying the series as cos x OR for making the correct connection to cos x

$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots + \frac{(-1)^n}{(2n)!}x^{2n} + \ldots$
$= \text{cosx}$

OR

$\text{xy'} + y = (\text{xy})' = (\text{sinx})' = \text{cosx}$

0	1	2	3	4

The student response earns four of the following points:

1 point is earned for the correct series for y' OR for the correct series for $xf(x)$

$y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \ldots + \frac{(-1)^n 2nx^{2n-1}}{(2n+1)!} + \ldots$

1 point is earned for the correct series for xy' OR for identifying the series as sin x

$\text{xy'} = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \ldots + \frac{(-1)^n 2nx^{2n}}{(2n+1)!} + \ldots$

OR

$\text{xy} = \text{xf}(x) = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \ldots$
$= \text{sinx}$

1 point is earned for the correct series for $xy' + y$ OR for handling $xy' + y$

$\text{xy'} + y = 1 - (\frac{2}{3!} + \frac{1}{3!})x^2 + (\frac{4}{5!} + \frac{1}{5!})x^4 - (\frac{6}{7!} + \frac{1}{7!})x^6 + \ldots$
$+ (-1)^n (\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!})x^{2n} + \ldots$

Series

1 point is earned for identifying the series as cos *x* OR for making the correct connection to cos *x*

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ... + \frac{(-1)^n}{(2n)!}x^{2n} + ...$$
$$= \cos x$$

OR

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ... + \frac{(-1)^n}{(2n)!}x^{2n} + ...$$
$$= \cos x$$

$$xy' + y = (xy)' = (\sin x)' = \cos x$$

23. For $x > 0$, the power series $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ... + (-1)^n \frac{x^{2n}}{(2n+1)!} + ...$ converges to which of the following?

(A) $\cos x$

(B) $\sin x$

(C) $\frac{\sin x}{x}$ ✓

(D) $e^x - e^{x^2}$

(E) $1 + e^x - e^{x^2}$

24. Which of the following is the Maclaurin series for e^{3x}?

Series

(A) $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(B) $3 + 9x + \frac{27x^2}{2} + \frac{81x^3}{3!} + \frac{243x^4}{4!} + \cdots$

(C) $1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \cdots$

(D) $1 + 3x + \frac{3x^2}{2} + \frac{3x^3}{3!} + \frac{3x^4}{4!} + \cdots$

(E) $1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \cdots$ ✓

25. The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\frac{x^2}{1-x^2}$?

(A) $1 + x^2 + x^4 + x^6 + x^8 + \cdots$

(B) $x^2 + x^3 + x^4 + x^5 + \cdots$

(C) $x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots$

(D) $x^2 + x^4 + x^6 + x^8 + \cdots$ ✓

(E) $x^2 - x^4 + x^6 - x^8 + \cdots$

26. What is the coefficient of x^2 in the Taylor series for $\frac{1}{(1+x)^2}$ about $x = 0$?

Series

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

(C) 1

(D) 3 ✓

(E) 6

27. The Maclaurin series for the function f is given by $f(x)=\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^{n}$. What is the value of $f(3)$?

(A) -3

(B) $-\frac{3}{7}$ ✓

(C) $\frac{4}{7}$

(D) $\frac{13}{16}$

(E) 4

Let f be a function that has derivatives of all orders for all real numbers. Assume $f(1)=3, f'(1)=-2, f''(1)=2$ and $f'''(1)=4$

28. Write the second-degree Taylor polynomial for f', the derivative of f, about $x=1$ and use it to approximate $f'(1.2)$.

Series

Please respond on separate paper, following directions from your teacher.

Part C

3 points are earned for the correct second degree Taylor polynomial about $x = 1$

< -1 > each incorrect term in polynomial

< -1 > error in approximation

$T'_3(x) = -2 + 2(x-1) + 2(x-1)^2$

$f'(1.2) \approx -2 + 0.4 + 0.08 = -1.52$

Note: < -1 > once for use of = rather than ≈

0	1	2	3 ✓

The student response earns three of the following points:

3 points are earned for the correct second degree Taylor polynomial about $x = 1$

< -1 > each incorrect term in polynomial

< -1 > error in approximation

$T'_3(x) = -2 + 2(x-1) + 2(x-1)^2$

$f'(1.2) \approx -2 + 0.4 + 0.08 = -1.52$

Note: < -1 > once for use of = rather than ≈

Let *f* be the function given by \(f(x)=e^{\frac{x}{2}}\).

29. Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x=0$.

Series

Please respond on separate paper, following directions from your teacher.

Part A

1 point is earned for first two terms

1 point is earned for second two terms

1 point is earned for general term

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$
$$e^{x/2} = 1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \cdots + \frac{(x/2)^n}{n!} + \cdots$$
$$= 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots$$

0	1	2	3 ✓

The student response earns three of the following points:

1 point is earned for first two terms

1 point is earned for second two terms

1 point is earned for general term

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$
$$e^{x/2} = 1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \cdots + \frac{(x/2)^n}{n!} + \cdots$$
$$= 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots$$