Function Compositions: Understanding Practice 1

Goal: Be able to justify how the domain and range change after a composition.

Consider the functions:

$$f: A \to B \quad \text{and} \quad g: C \to D$$

Where $B \neq C$ and we want to look at the composition $g \circ f = h$

Case 1 Practice: $B \subset C$ (B is a subset of C)

1. Use pictures to justify why the domain of h is A and the range is a subset of D.

2. Consider an example where $g: \mathbb{R} \to \mathbb{R}$, for this example we want a function whose range is a subset of the domain of g (which is $\mathbb{R}$). Let's use $f: \mathbb{R} \to [1, \infty)$. Note that $[1, \infty) \subset \mathbb{R}$

   If $g: x \mapsto 3x - 5$ what is the domain and range of h?

3. Why do we not need to know what f does explicitly to determine the domain and range of h?

4. Let's do another example, $g: \mathbb{R} \to \mathbb{R}$ where $g: x \mapsto 2x^3 + 1$ and this time $f: [0, \infty) \to (0,1]$. What is the domain and range of h?
   (The domain of f is $\{x \mid x \geq 0\}$, the range is $\{y \mid 0 < y \leq 1\}$)

5. Make an example of two functions f and g that fit in this case. Use mapping notation to describe the functions and then determine the domain and range of h.

Case 2 Practice: $C \subset B$ (C is a subset of B)

6. Use pictures to justify why the range of h is D and the domain is a subset of A.

7. Let's start with an easy example of $f: \mathbb{R} \to \mathbb{R}$ and now we need a function whose domain is a subset of the range of f (a subset of $\mathbb{R}$). Let's just say that $g: [0, \infty) \to \mathbb{R}$.

   If $f: x \mapsto -\frac{1}{2}x + 6$ then what is the domain and range of h?

8. Again, let's define a function $f: \mathbb{R} \to \mathbb{R}$ and make $g: [0, 10) \to \{0,1\}$. If $f: x \mapsto \frac{1}{x}, x \neq 0$ and $f: 0 \mapsto 0$ then what is the domain and range of h?

9. Make an example of two function f and g that fit in this case. Use mapping notation to describe the functions and then determine the domain and range of h.

Case 3: Neither B nor C is a subset of the other.

10. Use pictures to justify why the range of h is a subset of D and the domain is a subset of A.

11. For an example we need f to have a restricted range and g to have a restricted domain where there is overlap between the two. Consider $f: x \mapsto x^2$ and $g: x \mapsto \sqrt{4-x}$. Here we have $f: \mathbb{R} \to [0, \infty)$ and $g: (-\infty, 4] \to [0, \infty)$.

    What is the intersection of the range of f and the domain of g?

    Use the intersection to determine the range of h.

    Use the intersection to determine the domain of h.

12. For another example, let's consider $f: \mathbb{R} \to [-2, \infty)$ and $g: (-\infty, 5) \to (0, \infty)$ where $f: x \mapsto |x| - 2$ and $g: x \mapsto \frac{1}{\sqrt{5-x}}$

    Determine the domain and range of h.

13. Make an example of two functions $f: A \to B$ and $g: C \to D$ such that the domain and range of $g \circ f$ is A and D respectively, but the domain of $f \circ g$ is just $\{1\}$ (the number 1) and the range is just $\{2\}$ (the number 2).