Function Compositions: Understanding Practice 1

Goal: Be able to justify how the domain and range change after a composition.

Consider the functions:

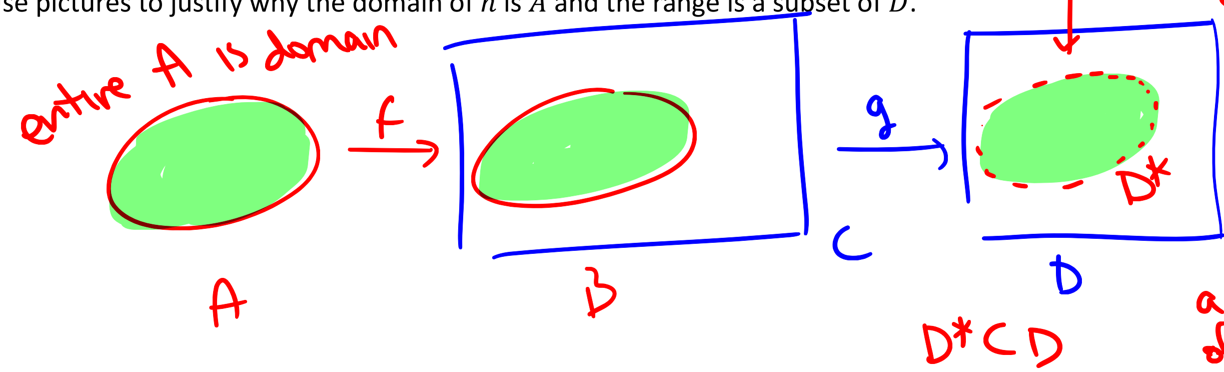
$$f: A \to B \quad \text{and} \quad g: C \to D$$

Where $B \neq C$ and we want to look at the composition $g \circ f = h$

Case 1 Practice: $B \subset C$ (B is a subset of C)

1. Use pictures to justify why the domain of h is A and the range is a subset of D.

Since we only give g a subset of its domain it only gives a subset of its range



2. Consider an example where $g: \mathbb{R} \to \mathbb{R}$, for this example we want a function whose range is a subset of the domain of g (which is $\mathbb{R}$). Let's use $f: \mathbb{R} \to [1, \infty)$. Note that $[1, \infty) \subset \mathbb{R}$

   If $g: x \mapsto 3x - 5$ what is the domain and range of h?

$f: \mathbb{R} \to [1, \infty)$ $\qquad$ $g: \mathbb{R} \to \mathbb{R}$

$x \mapsto 3x-5$

we give g values that are ≥ 1

$\Rightarrow$ $x \geq 1$

$3x \geq 3$

$\underbrace{3x-5}_{g(x)} \geq -2$

$g(x) \geq -2$

range

Domain is still going to be $\mathbb{R}$

Range is $[-2, \infty)$

3. Why do we not need to know what f does explicitly to determine the domain and range of h?

B/C it only matters what f gives to g. Since we know all the values f gives are okay we can ~~f~~focus on how g might only give a subset of its values.

4. Let's do another example, $g: \mathbb{R} \to \mathbb{R}$ where $g: x \mapsto 2x^3 + 1$ and this time $f: [0, \infty) \to (0,1]$. What is the domain and range of h?
(The domain of f is $\{x \mid x \geq 0\}$, the range is $\{y \mid 0 < y \leq 1\}$)

$f: [0,\infty) \to (0,1]$ $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x^3+1$

this is given to g

Domain is unchanged so still $[0,\infty)$

$x \in (0,1] \Rightarrow 0 < x \leq 1$

$0 < x^3 \leq 1$

$0 < 2x^3 \leq 2$

$1 < 2x^3+1 \leq 3$

$1 < g(x) \leq 3$

New range is $(1,3]$

5. Make an example of two functions f and g that fit in this case. Use mapping notation to describe the functions and then determine the domain and range of h.

$f: A \to B$ $g: \mathbb{R} \to C$, $x \mapsto g(x)$ Anything like this

say $f: \mathbb{Z} \to \{0,1\}$ $g: \mathbb{R} \to (0,1]$

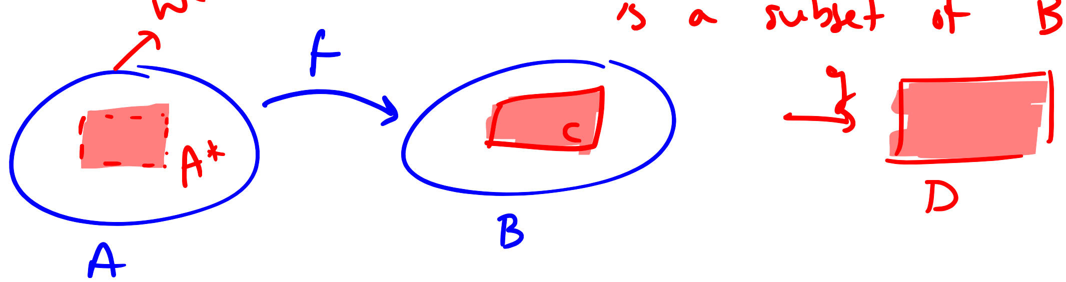
f: even $\mapsto 0$, odd $\mapsto 1$ $g: x \mapsto \frac{1}{x^2+1}$

domain of $g \circ f$ is $\mathbb{Z}$ and range is $g(\{0,1\}) = \{1, \frac{1}{2}\}$

Case 2 Practice: $C \subset B$ (C is a subset of B)

6. Use pictures to justify why the range of h is D and the domain is a subset of A.

we must give f a subset of A so the range is a subset of B.



7. Let's start with an easy example of $f: \mathbb{R} \to \mathbb{R}$ and now we need a function whose domain is a subset of the range of f (a subset of $\mathbb{R}$). Let's just say that $g: [0, \infty) \to \mathbb{R}$.

   If $f: x \mapsto -\frac{1}{2}x + 6$ then what is the domain and range of h?

$f: \mathbb{R} \to \mathbb{R}$ $\quad g: [0, \infty) \to \mathbb{R}$

$x \mapsto -\frac{1}{2}x + 6$

range will stay $\mathbb{R}$

need $f(A^*) = [0, \infty)$

$\Rightarrow$ if $x \in A^*$ then $f(x) \in [0, \infty)$

$\Rightarrow -\frac{1}{2}x + 6 \geq 0$

$\Rightarrow \frac{1}{2}x \leq 6$

$x \leq 12 \to$ New domain

$g \circ f: (-\infty, 12] \to \mathbb{R}$

8. Again, let's define a function $f: \mathbb{R} \to \mathbb{R}$ and make $g: [0, 10) \to \{0,1\}$. If $f: x \mapsto \frac{1}{x}, x \neq 0$ and $f: 0 \mapsto 0$ then what is the domain and range of h?

$f: \mathbb{R} \to \mathbb{R}$ $\quad g: [0, 10) \to \{0, 1\}$ $\to$ Range is still $\{0, 1\}$

$x \mapsto \frac{1}{x}$

$0 \mapsto 0$

we need $f(A^*) = [0, 10)$

$\Rightarrow f(x) \in [0, 10)$

$\Rightarrow 0 \leq \frac{1}{x} < 10$ $\quad$ or $\quad 0 \leq 0 < 10$ ✓

$0 \leq 1 < 10x$ $\quad x = 0$

$x > \frac{1}{10}$

$\boxed{\text{Domain} = \{0\} \cup (\tfrac{1}{10}, \infty)}$

9. Make an example of two function f and g that fit in this case. Use mapping notation to describe the functions and then determine the domain and range of h.

$f: \mathbb{R} \to \mathbb{R}$ $\quad g: B \to C$

Range is C

Domain is A^*, $f(A^*) = B$

ex) $f: \mathbb{R} \to \mathbb{R}$ $\quad g: [0, \infty) \to [0, \infty)$ $\to$ Range will be this

$x \mapsto \frac{5x - 20}{3}$ $\quad x \mapsto \sqrt{x}$

find A^* so $f(A^*) = [0, \infty) \Rightarrow$

$f(x) \geq 0$

$\frac{5x - 20}{3} \geq 0$

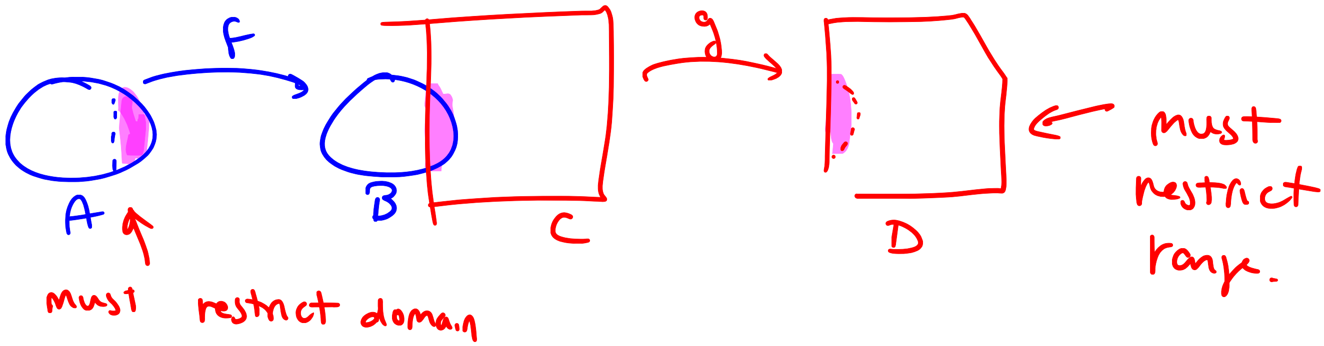
$5x - 20 \geq 0$ $\quad$ New domain

$5x \geq 20 \to x \geq 4$

$\Rightarrow g \circ f: [4, \infty) \to [0, \infty)$

Case 3: Neither B nor C is a subset of the other.

10. Use pictures to justify why the range of h is a subset of D and the domain is a subset of A.



11. For an example we need f to have a restricted range and g to have a restricted domain where there is overlap between the two. Consider $f: x \mapsto x^2$ and $g: x \mapsto \sqrt{4-x}$. Here we have $f: \mathbb{R} \to [0, \infty)$ and $g: (-\infty, 4] \to [0, \infty)$.

What is the intersection of the range of f and the domain of g?

$[0,\infty) \cap (-\infty,4] = [0,4]$

Use the intersection to determine the range of h.

Range is what g takes the intersection to

$g([0,4]) = D^* \Rightarrow$ if $x \in [0,4]$ then $g(x) \in D^*$

$0 \le x \le 4 \Rightarrow 0 \ge -x \ge -4 \Rightarrow 4 \ge 4-x \ge 0$

$\Rightarrow 2 \ge \sqrt{4-x} \ge 0$

$2 \ge g(x) \ge 0 \Rightarrow$ Range is $[0,2]$

Use the intersection to determine the domain of h.

Domain is what will be sent to $[0,4]$ under f.

if $x \in A^* \Rightarrow f(x) \in [0,4]$

$\Rightarrow 0 \le x^2 \le 4$

$\Rightarrow 0 \le |x| \le 2$

$\Rightarrow -2 \le x \le 2$

New Domain is $[-2,2]$

12. For another example, let's consider $f: \mathbb{R} \to [-2, \infty)$ and $g: (-\infty, 5) \to (0, \infty)$ where $f: x \mapsto |x| - 2$ and $g: x \mapsto \frac{1}{\sqrt{5-x}}$

Determine the domain and range of h.

$f: \mathbb{R} \to [-2, \infty)$ $\qquad$ $g: (-\infty, 5) \to (0, \infty)$

$x \mapsto |x| - 2$ $\qquad$ $x \mapsto \frac{1}{\sqrt{5-x}}$

$\Rightarrow$ intersection in the middle is $[-2, 5)$

domain A^* s.t if $x \in A^* \Rightarrow f(x) \in [-2, 5)$

$\Rightarrow -2 \leq |x| - 2 < 5$

$0 \leq |x| < 7 \Rightarrow -7 < x < 7$

$\Rightarrow$ new domain $A^* = (-7, 7)$

Range D^* s.t if $x \in [-2, 5)$ then $g(x) \in D^*$

$\Rightarrow -2 \leq x < 5$

$\Rightarrow 2 \geq -x > -5$

$\Rightarrow 7 \geq 5 - x > 0$

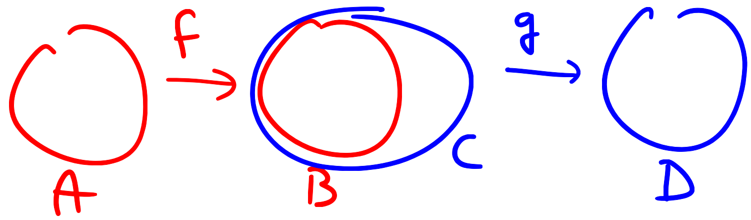
$\Rightarrow \frac{7}{5-x} \geq 1 > 0$

$\Rightarrow \frac{1}{5-x} \geq \frac{1}{7} > 0 \Rightarrow g(x) \geq \sqrt{\frac{1}{7}}$

New Range is $\left[\sqrt{\frac{1}{7}}, \infty\right)$

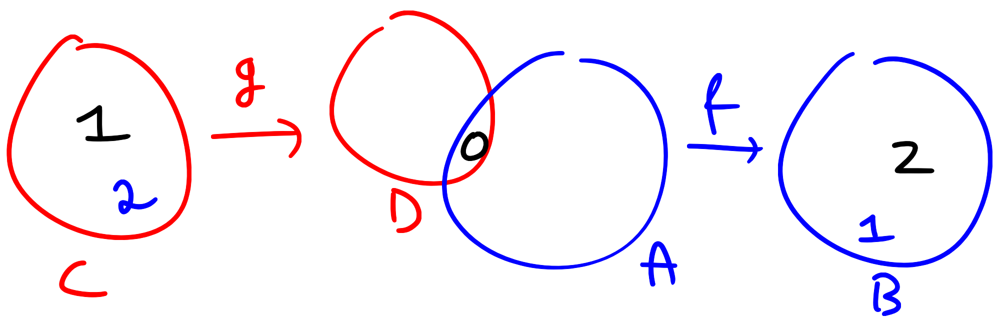
13. Make an example of two functions $f: A \to B$ and $g: C \to D$ such that the domain and range of $g \circ f$ is A and D respectively, but the domain of $f \circ g$ is just $\{1\}$ (the number 1) and the range is just $\{2\}$ (the number 2).

if f is done first then there is no restrictions



$\Rightarrow \boxed{B = C}$

if we do g first then domain is $\{1\}$ and range is $\{2\}$



$\Rightarrow 1 \in C$ and $2 \in B$ (but in both since $C = B$)

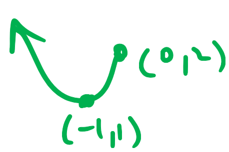
$\Rightarrow D \cap A$ is only 1 element, say $\{0\} = A \cap D$

↳ lets make $D = [0, \infty)$ and $A = (-\infty, 0]$

↳ Im lazy and just going to make $B = C = [1, \infty)$

so $g: [1, \infty) \to [0, \infty)$ and $f: (-\infty, 0] \to [1, \infty)$

$x \mapsto \sqrt{x-1}$ $\qquad$ $x \mapsto (x+1)^2 + 1$



★ $g(1) = 0$ $\qquad$ ★ $f(0) = 2$