Function Transformations and Inverses: Understanding Practice 2

Goal: Apply transformations in any order into mapping notation or function notation and consider transformations of the inverse function.

When we apply transformations, we apply it in order of operations. If we stretch space and then shift it we would get the map

$$T: (x, y) \mapsto (bx + c, ay + d)$$

And in function notation:

$$g(x) = a \cdot f\left(\frac{1}{b}(x - c)\right) + d$$

In the map, the first operation is multiplication (stretch) and then addition (shift), in the function notation vertically it is the same, but horizontally it is the inverse: First **subtraction**, then **division**.

1. Write the map for the reverse order where we shift (by c and d) and then we stretch (by a and b) as before.

$$T_1: (x,y) \mapsto (b(x+c), a(y+d))$$

Note

$$f: x \mapsto y = f(x)$$

$$g_1: b(x+c) \mapsto a(y+d) = g(b(x+c)) = a(f(x)+d)$$

2. Put this into function notation.

from above $g(b(x+c)) = a(f(x)+d)$

let $b(x+c) = X \rightarrow x = \frac{X}{b} - c$

$$\Rightarrow g(X) = a\left[f\left(\frac{1}{b}X - c\right) + d\right]$$

3. Why are these not the same?

Because the order of transformations is dependent on the order of operations we should not expect them to be the same

4. Why can the order of reflection and stretch be interchanged and not affect the final transformation?

let $R: (x,y) \mapsto (-x, y)$

$S: (x,y) \mapsto (bx, y)$

$\Rightarrow T = R \circ S = S \circ R$

$R \circ S: (x,y) \mapsto (bx, y) \mapsto (-bx, y)$

$S \circ R: (x,y) \mapsto (-x, y) \mapsto (b(-x), y)$

same

Because they are both multiplication. They are associative operations

$-bx = -(bx) = b(-x)$

Consider the following list of transformations

A. Expand space vertically by 3
B. Shift Down by 5
C. Reflect over the x-axis
D. Shift Up by 1
E. Compress space vertically by 2

When I refer to transformation A I am referring to $A: (x, y) \mapsto (x, 3y)$, and the rest likewise.

5. Consider the composition of transformations, first A, then B, then C. Write this as T_1 as a composition of A, B, C

$T_1 = C \circ B \circ A$

$A: (x,y) \mapsto (x, 3y)$

$B: (x,y) \mapsto (x, y-5)$

$C: (x,y) \mapsto (x, -y)$

6. Write T_1 as a map and then in function notation.

$T_1: (x,y) \overset{A}{\longmapsto} (x, 3y) \overset{B}{\longmapsto} (x, 3y-5) \overset{C}{\longmapsto} (x, -(3y-5))$

$\Rightarrow T_1: (x,y) \longmapsto (x, -(3y-5))$

$g(x) = -[3f(x) - 5]$

7. Simplify the map of T_1 into the form $y \mapsto ay + d$ to identify the standard stretch then shift transformations that occurred.

we have $y \mapsto -(3y-5) = -3y+5$

so we actually could have expanded vert by 3 Then reflected over x-axis and then shifted up 5

8. State the mapping and function notation of the transformation $T_2 = E \circ D \circ C \circ B \circ A$ and then simplify it into the standard stretch then shift form.

$T_2 : (x,y) \xmapsto{C \cdot B \cdot A} (x, -(3y-5)) \xmapsto{D} (x, -(3y-5)+1) \xmapsto{E} \left(x, \frac{-(3y-5)+1}{2}\right)$

$\Rightarrow T_2 : (x,y) \mapsto \left(x, \frac{-(3y-5)+1}{2}\right)$

$\Rightarrow g_2(x) = \frac{1}{2}\left[-\left(3f(x)-5\right)+1\right] = -\frac{3}{2}f(x) + 3$

so just expand vert by $3/2$, reflect over x, up 3

9. If T_2 happened **horizontally** (left/right and over the y-axis) what is the mapping and function notation of it then? **Remember that everything in function notation horizontally gets reversed. You should be able to simplify it and get the same result as above.

→ same as vertical

$T_2 : (x,y) \longmapsto \left(\frac{-(3x-5)+1}{2}, y\right)$

$f: x \mapsto y$

$g: \underbrace{\frac{1-(3x-5)}{2}}_{X} \mapsto y = f(x) = g(X) \Rightarrow g(X) = f\left(-\frac{2}{3}X + 2\right)$

$= f\left(-\frac{2}{3}(x-3)\right)$

$X = \frac{1-(3x-5)}{2}$

$\Rightarrow x = \frac{-2X+1+5}{3} = -\frac{2}{3}X + 2$

* Note could simplify earlier

$T_2 : (x,y) \mapsto \left(-\frac{3}{2}x+3, y\right)$

$\Rightarrow$ expand by $3/2$, reflect, right 3

$\Rightarrow g(x) = f\left(-\frac{2}{3}(x-3)\right)$

10. Why can you swap a horizontal transformation and ~~if~~ a vertical transformation without changing the overall transformation?

$$H : (x, y) \mapsto (bx + c, y)$$
$$V : (x, y) \mapsto (x, ay + d)$$

$$H \circ V : (x,y) \xmapsto{V} (x, ay+d) \xmapsto{H} (bx+c, ay+d)$$

$$V \circ H : (x,y) \xmapsto{H} (bx+c, y) \xmapsto{V} (bx+c, ay+d)$$

a horizontal transformation does not change anything vertical. and vice versa

11. Apply the **horizontal** transformations $T_3 = A \circ B \circ C \circ D \circ E \circ B \circ C$ put it in mapping and function form and simplify to a standard stretch and then shift form.

$$T_3 : (x,y) \xmapsto{C} (-x, y) \xmapsto{B} (-x-5, y) \xmapsto{E} \left(\frac{-x-5}{2}, y\right) \xmapsto{D} \left(\frac{-x-5}{2} + 1, y\right)$$

$\cancel{C}\cancel{B}\cancel{E}\cancel{D}\cancel{C}BA$

$$\therefore \left(\frac{-x-5}{2} + 1, y\right) \xmapsto{C} \left(\frac{x+5}{2} - 1, y\right) \xmapsto{B} \left(\frac{x+5}{2} - 1 - 5, y\right)$$

$$\therefore \left(\frac{x+5}{2} - 6, y\right) \xmapsto{A} \left(\frac{3}{2}(x+5) - 18, y\right)$$

$$\Rightarrow T_3 : (x,y) \mapsto \left(\frac{3}{2}x - \frac{21}{2}\right) \quad \Rightarrow \text{expand horiz. by } \frac{3}{2} \text{ then left } \frac{21}{2}$$

$$g(x) = f\left(\frac{2}{3}\left(x + \frac{21}{2}\right)\right) = f\left(\frac{2}{3}x + 7\right)$$

When we invert the x, y values get swapped with the transformation:

$$I: (x, y) \mapsto (y, x)$$

And consider the transformation:

$$T: (x, y) \mapsto (x + 3, 2y)$$

After this transformation we get that

$$g(x) = 2f(x - 3)$$

12. Apply the transformation $T_4 = T \circ I$ and reason why

$$h(x) = 2f^{-1}(x - 3)$$

$T_4: (x,y) \overset{\text{invert}}{\longmapsto} (y,x) \overset{\text{transform}}{\longmapsto} (y+3, 2x)$

$\Rightarrow$ we have transformed f^{-1} by right 3 and vert. exp by 2

$f: x \mapsto y = f(x) \Rightarrow x = f^{-1}(y)$

$h: y+3 \mapsto 2x = h(y+3) = 2f^{-1}(y)$

let $Y = y+3 \Rightarrow h(Y) = 2f^{-1}(Y-3)$

$\rightarrow y = Y - 3$

$h(x) = 2f^{-1}(x-3)$

13. Apply the transformation $T_5 = I \circ T$ and reason why

$$k(x) = f^{-1}\left(\frac{1}{2}x\right) + 3$$

$T_5: (x,y) \overset{\text{transform}}{\longmapsto} (x+3, 2y) \overset{\text{invert}}{\longmapsto} (2y, x+3)$

we invert the transformation so should be g^{-1}

$f: x \mapsto y = f(x) \Rightarrow f^{-1}(y) = x$

$k: 2y \mapsto x+3 = k(2y)$

$f^{-1}(y) + 3 = k(2y)$

let $Y = 2y \Rightarrow k(Y) = f^{-1}\left(\frac{Y}{2}\right) + 3$

$\frac{Y}{2} = y$

$k(x) = f^{-1}\left(\frac{x}{2}\right) + 3$

Note

$g(x) = 2f(x-3)$

$\Rightarrow x = 2f(y-3)$

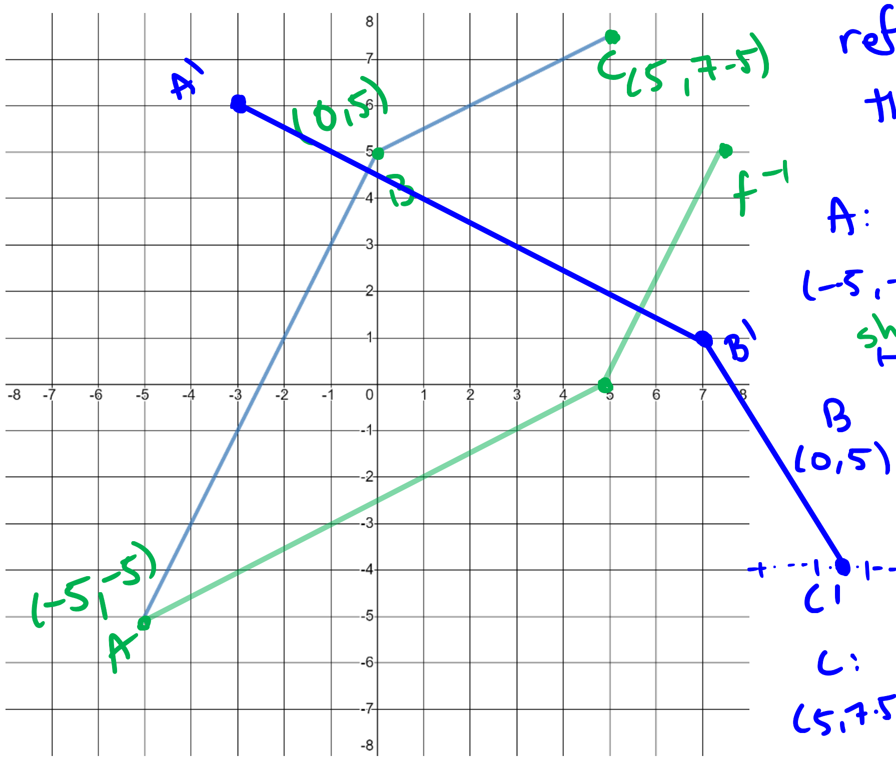
$\frac{1}{2}x = f(y-3)$

$f^{-1}\left(\frac{1}{2}x\right) + 3 = y = g^{-1}(x)$

14. Given the following graph of f, graph the image of

$$-f^{-1}(x-2)+1$$

take inverse, then reflect over x-axis, then right 2 and up 1



A:

$(-5,-5) \xmapsto{\text{invrt}} (-5,-5) \xmapsto{\text{reflect}} (-5,5)$

$\xmapsto{\text{shift}} (-3,6)$

B

$(0,5) \mapsto (5,0) \mapsto (5,0) \mapsto (7,1)$

C:

$(5,7.5) \mapsto (7.5,5) \mapsto (7.5,-5)$

$\mapsto (9.5,-4)$

15. Given the following graph of f, graph the image of

$$\frac{1}{3}f^{-1}(1-2x) = \frac{1}{3}f^{-1}\left(-2\left(x-\frac{1}{2}\right)\right)$$

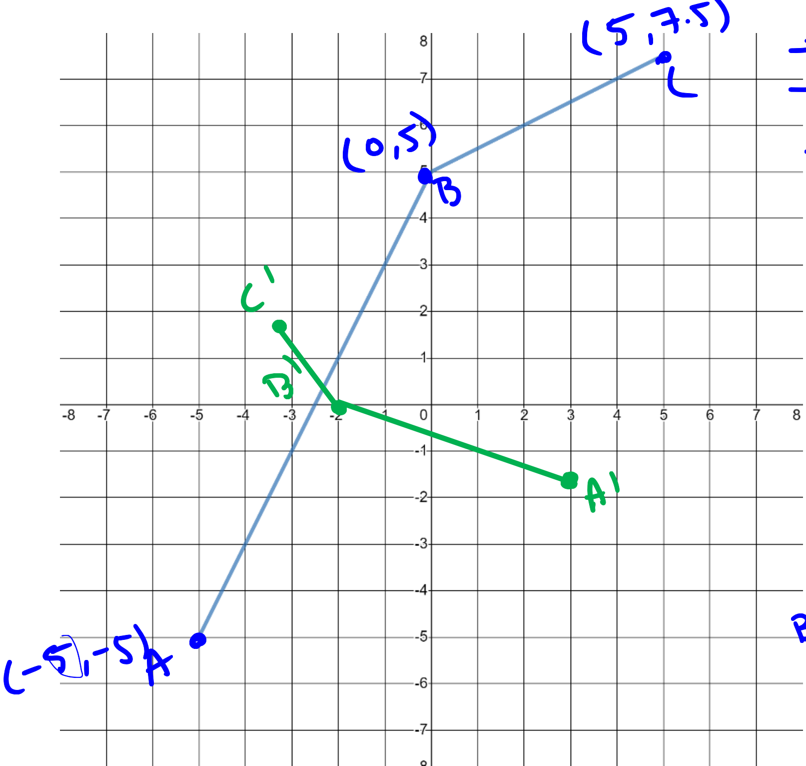
→ invert

→ vert comp by 3

→ horiz comp by 2

→ reflect over y-axis

→ right 0.5



A:

$(-5,-5) \xmapsto{\text{invrt}} (-5,-5) \xmapsto{\text{comp.}} \left(-\frac{5}{2}, -\frac{5}{3}\right)$

$\xmapsto{\text{reflect}} \left(\frac{5}{2}, -\frac{5}{3}\right) \xmapsto{\text{shift}} \left(3, -5/3\right)$

B: $(0,5) \mapsto (5,0) \mapsto \left(\frac{5}{2}, 0\right) \mapsto \left(-\frac{5}{2}, 0\right)$

$\mapsto (-2,0)$

C: $(5,7.5) \mapsto (7.5,5) \mapsto \left(\frac{7.5}{2}, \frac{5}{3}\right)$

$\mapsto \left(-\frac{7.5}{2}, 5/3\right) \mapsto \left(-3.25, 5/3\right)$