Function Transformations and Inverses: Understanding Practice 2

Goal: Apply transformations in any order into mapping notation or function notation and consider transformations of the inverse function.

When we apply transformations, we apply it in order of operations. If we stretch space and then shift it we would get the map

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$
$$T: (x, y) \mapsto (bx + c, ay + d)$$

And in function notation:

$$g(x) = a \cdot f\left(\frac{1}{b}(x - c)\right) + d$$

In the map, the first operation is multiplication (stretch) and then addition (shift), in the function notation vertically it is the same, but horizontally it is the inverse: First **subtraction**, then **division**.

1. Write the map for the reverse order where we shift (by c and d) and then we stretch (by a and b) as before.

$T: (x, y) \mapsto (b(x+c), a(y+d))$

$f: x \mapsto y = f(x)$

$g: b(x+c) \mapsto a(y+d) = g(b(x+c)) = a(f(x) + d)$

2. Put this into function notation.

$g(b(x+c)) = a(f(x) + d)$

Let $b(x+c) = X$; $\boxed{x} = \frac{\boxed{X}}{b} - c$ x after transformation (g)

original x in f

$g(X) = a[f(\frac{1}{b}x - c) + d]$

3. Why does it make sense that these are not the same?

Because the order of transformation depends on the order of operation. In this case, we are NOT stretching x and y, but (x+c), (y+d) instead. Thus, we should not expect them to be the same.

4. Why can the order of reflection and stretch be interchanged and not affect the final transformation?

Let $R: (x,y) \mapsto (-x, y)$

$S: (x,y) \mapsto (bx, y)$

$\Rightarrow T = R \circ S = S \circ R$

$R \circ S: (x,y) \mapsto (bx, y) \mapsto (-(bx), y)$

same!

$S \circ R: (x,y) \mapsto (-x, y) \mapsto (b(-x), y)$

Since both of them are multiplication, they are associative operations.

$-bx = -(bx) = b(-x)$

Consider the following list of transformations

A. Expand space vertically by 3
B. Shift Down by 5
C. Reflect over the x-axis
D. Shift Up by 1
E. Compress space vertically by 2

When I refer to transformation A I am referring to $A: (x, y) \mapsto (x, 3y)$, and the rest likewise.

5. Consider the composition of transformations, first A, then B, then C. Write this as T_1 as a composition of A, B, C

$T_1 = C \circ B \circ A$

$A: (x, y) \mapsto (x, 3y)$

$B: (x, y) \mapsto (x, y-5)$

$C: (x, y) \mapsto (x, -y)$

6. Write T_1 as a map and then in function notation.

$T_1: (x,y) \xrightarrow{A} (x, 3y) \xrightarrow{B} (x, 3y-5) \xrightarrow{C} (x, -(3y-5))$

$\Rightarrow T_1: (x,y) \mapsto (x, -(3y-5))$

$g(x) = -[3f(x) - 5]$

7. Simplify the map of T_1 into the form $(x, y) \mapsto (x, ay + d)$ to identify the standard stretch then shift transformations that occurred.

$y \mapsto -(3y-5) = -3y+5$

① Expand vertically by 3

② Reflect over x-axis

③ Shift up 5

8. State the mapping and function notation of the transformation $T_2 = E \circ D \circ C \circ B \circ A$ and then simplify it into the standard stretch then shift form.

$T_2 : (x, y) \xmapsto{C \circ B \circ A} (x, -(3y-5)) \xmapsto{D} (x, -(3y-5)+1) \xmapsto{E} \left(x, \frac{-(3y-5)+1}{2}\right)$

$\Rightarrow T_2 : (x, y) \mapsto \left(x, \frac{-(3y-5)+1}{2}\right)$

$\Rightarrow g_2(x) = \frac{1}{2}\left[-(3f(x)-5)+1\right] = -\frac{3}{2}f(x)+3$

① Expand vertically by $\frac{3}{2}$

② Reflect over x

③ Up 3

9. If T_2 happened **horizontally** (left/right and over the y-axis) what is the mapping and function notation of it then? **Remember that everything in function notation horizontally gets reversed. You should be able to simplify it and get the same result as above.

METHOD 1: Mapping → Function

$T_2 : (x, y) \longmapsto \left(\frac{-(3x-5)+1}{2}, y\right)$ same as vertical

$f : x \mapsto y$

$g : \underbrace{\frac{1-(3x-5)}{2}}_{X} \mapsto y = f(x) = g(X)$

$X = \frac{1-(3x-5)}{2}$

$x = \frac{-2X+1+5}{2} = -\frac{2}{3}X + 2$

$\Rightarrow g(X) = f(-\frac{2}{3}X + 2)$

$= f(-\frac{2}{3}(X-3))$

NOTE: could be simplified earlier

$T_2 : (x, y) \mapsto (-\frac{3}{2}x+3, y)$

$\Rightarrow$ Expand by $\frac{3}{2}$, reflect, right by 3

$\Rightarrow g(x) = f(-\frac{2}{3}(x-3))$

METHOD 2: Transformation using function notation

apply on x only, not $\frac{1}{3}x$

$y = f(x) \rightarrow f(\frac{1}{3}x) \rightarrow f(\frac{1}{3}(x+5)) \rightarrow f(\frac{1}{3}(-x+5)) \rightarrow f(\frac{1}{3}(-(x-1)+5))$

$\rightarrow f(\frac{1}{3}(-(2x-1))+5) = f(-\frac{2}{3}(x-3))$

They are the same thing!