Polynomials: Understanding Practice 3

Goal: Build a better understanding of the graphs of polynomials on a very large scale and a small scale to prepare for calculus. Use remainder theorem and factor theorem with polynomials in $\mathbb{Z}[X]$

When we look at a polynomial

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

For $x \gg 0$ (this means significantly greater than 0) the polynomial will be approximate to the leading term.

$$p(x) \approx a_n x^n$$

This means that $\frac{p(x)}{a_n x^n} \approx 1$ for large x.

1. Consider the polynomial $p(x) = -3x^4 + 50x^2 - 100x + 500$. In desmos or geogebra, write the quotient of $p(x)$ and $-3x^4$ as a function Q and use it to make a table of values for the quotient.

x	1	10	100	1000	10^6
$\frac{p(x)}{-3x^4}$	-149	0.85	0.998365	0.999933..	0.9999...

x	-1	-10	-100	-1000	-10^6
$\frac{p(x)}{-3x^4}$	$-215.\overline{6}$	$0.78\overline{3}$..	0.99829..	0.999983..	0.9999...

How does the table and the graph show that as $|x| \to \infty$ we have that $p(x) \to -3x^4$

as $|x| \to \infty$ (gets big) we have $\frac{p(x)}{-3x^4} \to 1$

so $p(x) \to -3x^4$ (if the quotient is 1 then top and bottom are the same)

2. What would the quotient be if we divided by $-x^4$ instead?

$\frac{p(x)}{-x^4} \longrightarrow 3$; can graph it and see a horizontal asymptote at $y=3$

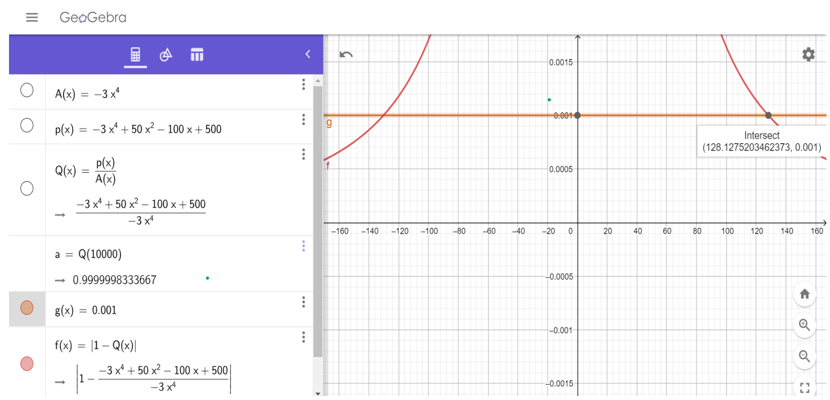
3. The percent error of p and the leading term is $\frac{|p(x)-a_n|}{a_n} = \left|1 - \frac{p(x)}{a_n}\right|$, use technology to determine the interval of x when the percent error is less than $0.1\ \%$

graph $f(x) = \left|1 - \frac{p(x)}{-3x^4}\right|$

ang $g(x) = 0.001$.

Intersect at $x = 128$ and $x = -130$

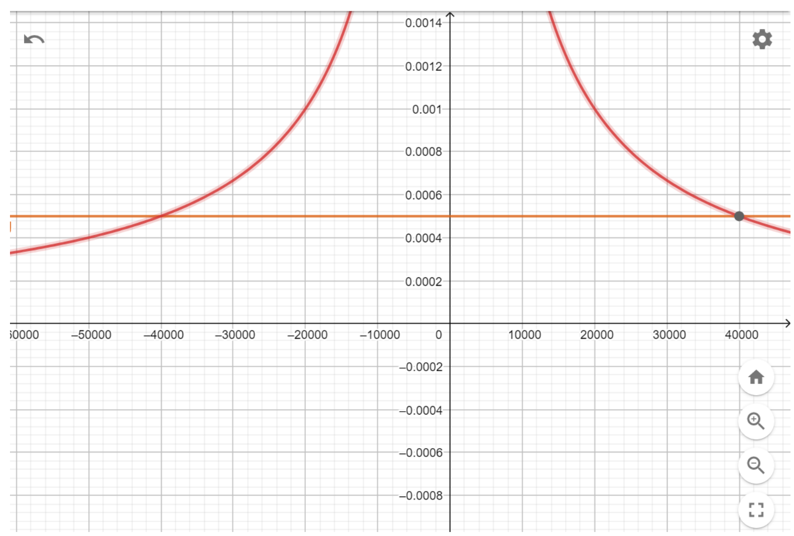
so want $\boxed{x > 128 \text{ or } x < -130}$



4. For large x, what does the polynomial $P(x) = \frac{1}{4}x^5 + 5x^4 - 30x^3 + 4000x$ approximate, and how large does x need to be for the percent error to be less than $0.05\ \%$

$P(x) \approx \frac{1}{4}x^5$ and the error is less than 0.05%.

when $x > 39\,994$ or $x < -40\,006$



5. Using technology, construct a polynomial that is within $1\ \%$ of $5x^3$ when $|x| > 100$ but the error is greater or equal to $1\ \%$ if $|x| \le 100$

used sliders a, b, c to play with $P(x) = 5x^3 + ax^2 + bx + c$ and watched the error at $x = \pm 100$ and ± 99

found $p(x) = 5x^3 - 5x^2 + \frac{1}{2}x + 58$

adjusted a first then b then c

For small values of x the smaller powers of x are what are important. When we look at a polynomial

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

For x small (this means x values close to 0), the polynomial will be approximate to the final terms.

$$p(x) \approx a_2 x^2 + a_1 x + a_0 = f(x)$$

Here I am including the final 3 terms since we know what parabolas look like fairly well and calling this parabola f.

This means that $p(x) - f(x) \approx 0$ for small x.

6. Consider the polynomial $p(x) = x^4 - 2x^3 - 2x^2 + x - 2$. Using technology graph the polynomial and the parabola $f(x) = -2x^2 + x - 2$. Complete the table of values to illustrate how close they are for small x

x	-5	-1	-0.1	-0.01	-0.001
$p(x)$	818	-2	-2.1179	-2.01019	-2.001002
$f(x)$	-57	-5	-2.12	-2.0102	-2.001002
$\|p(x) - f(x)\|$	875	3	0.0021	2×10^{-6}	2×10^{-9}

x	0.001	0.01	0.1	1	5
$p(x)$	-1.999002	-1.990202	-1.9219	-4	328
$f(x)$	-1.999002	-1.9902	-1.92	-3	-47
$\|p(x) - f(x)\|$	2×10^{-9}	2×10^{-6}	0.0019	1	375

7. Why do you think we are looking at the difference here rather than the quotient like we did for large x?

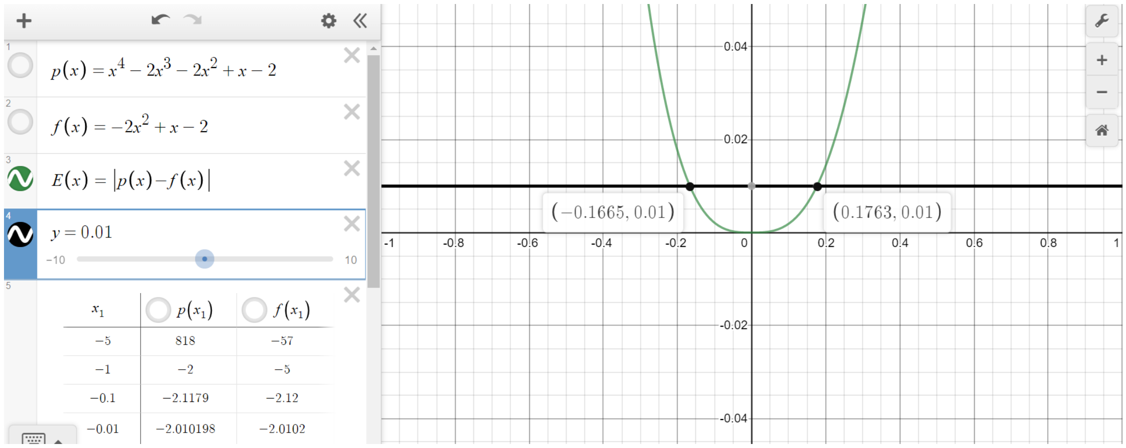
For large x $|p(x) - a_n x^n|$ doesn't go to 0 as x gets large.

For small x $\frac{p(x)}{f(x)}$ could be $\frac{0}{0}$

8. For the polynomial is the previous question, use technology to determine an interval of x such that $|p(x) - f(x)| < 0.01$

find that error < 0.01 when

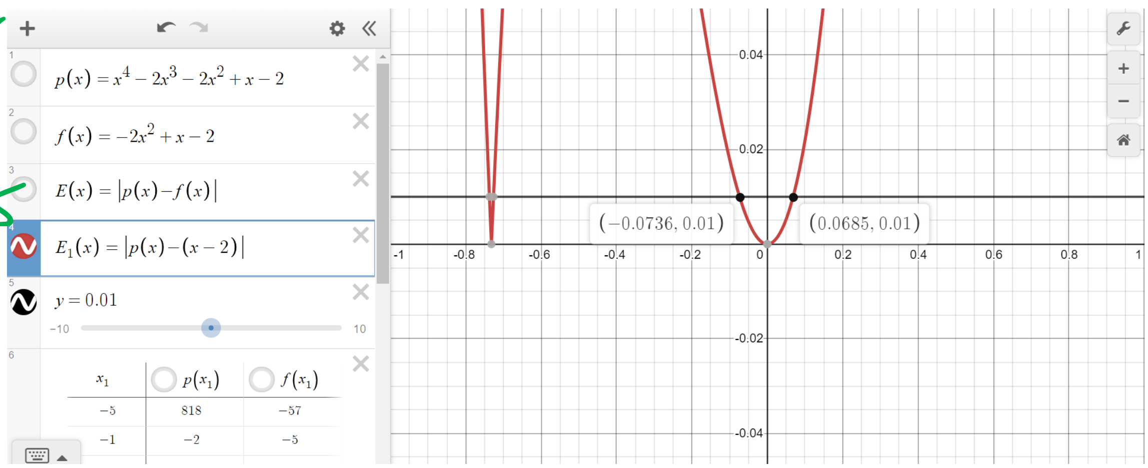
$-0.1665 < x < 0.1763$



9. If we only approximated $p(x)$ with the last two terms $y = x - 2$, what interval of x would give us $|p(x) - y| < 0.01$?

a much smaller interval, only

$-0.0736 < x < 0.0685$



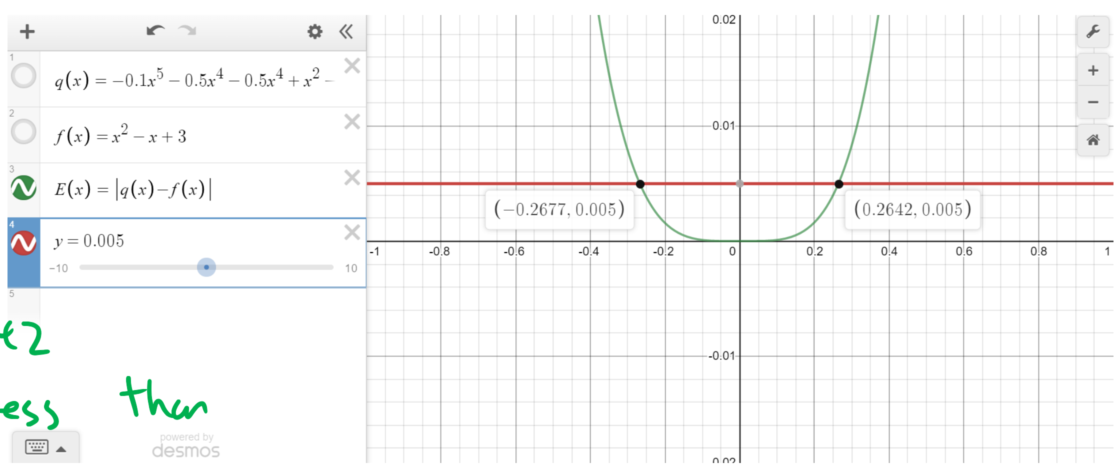
10. Given the polynomial $q(x) = -0.1x^5 - 0.5x^4 + x^2 - x + 3$ find a parabola that approximates q for small x and determine an interval of x such that the difference between them is less than 0.005

$q(x) \sim x^2 - x + 3$ for small x

on the interval

$-0.2677 < x < 0.2642$

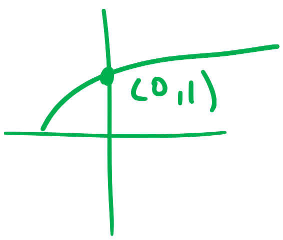
the error is less then 0.005



11. Graph the function $f(x) = \sqrt{x+1}$ and use technology to help build a parabola $ax^2 + bx + c$ that looks like f for small x. Discuss why you chose the values of $a, b,$ and c when building it. Use your polynomial to estimate $\sqrt{1.1}$

$f(x) = \sqrt{x+1}$

I got $\sqrt{x+1} \sim -0.2x^2 + \frac{1}{2}x + 1$



$ax^2 + bx + 1$ ← y-int (1)

slope (2)

what looks best (3)

but something -ve

from this

$x = 0.1$ ☆

$\sqrt{1.1} \sim -0.2(0.1)^2 + \frac{1}{2}(0.1) + 1$

$= -0.2(0.01) + 1.05$

$= -0.002 + 1.05$

$= 1.0498$

Remainder theorem says if $p(x)$ is divided by $x - a$ then its remainder is just $p(a)$. That is $p(a) = r$.

12. If the polynomial $p(x)$ is divided by $x - 3$ and the remainder is -2, what point must be on the curve p?

$p(3) = -2 \Rightarrow (3, -2)$

13. If the polynomial $p(x)$ is divisible by $x + 4$, what point must be on the curve p?

$p(-4) = 0 \Rightarrow (-4, 0)$

14. If we know $p(x)$ is divisible by $x + 4$ then what does its factored form look like?

$p(x) = (x+4)\, q(x)$

15. If a polynomial is divisible by $x+1$, but it has a remainder of 3 when divided by $x-1$ and a remainder of -1 when divided by $x+3$, what points must the curve pass through and how can we write it in factored form?

$p(-1)=0$
$p(1)=3$
$p(-3)=-1$

$\Rightarrow (-1,0)\,;\ (1,3)\,;\ (-3,-1)$

and $p(x) = (x+1)\,q(x)$

16. For the above polynomial, if it is a parabola what would the equation to the parabola be? Express the polynomial as $A \cdot q(x)$ where $q(x) \in \mathbb{Z}[X]$ and $A \in \mathbb{Q}$.

$p(x) = ax^2+bx+c = (x+1)(Mx+B)$ $p(1)=3$, $p(-3)=-1$

$\Rightarrow 3 = p(1) = 2(M+B)$

and $-1 = p(-3) = -2(-3M+B)$

$\Rightarrow \frac{3}{2} = M+B$ ①
$\frac{1}{2} = -3M+B$ ②

$\Rightarrow$ ① − ② : $1 = 4M$ $\Rightarrow M = 1/4\,;\ B = \frac{5}{4}$

$p(x) = (x+1)(\frac{1}{4}x + 5/4)$
$= \frac{1}{4}(x+1)(x+5)$

17. What if the polynomial was a cubic instead? Find one such cubic and explain why there are infinitely many cubic polynomials that satisfy these conditions.

if $p(x) = (x+1)(ax^2+bx+c)$ $p(1)=3$ $p(-3)=-1$

① $3 = 2(a+b+c)$
② $-1 = -2(9a-2b+c)$

$\Rightarrow$ 2 eqns with 3 unknowns $\Rightarrow \infty$ many solutions.

set $b=0$ (or anything) and find a and c

① $\frac{3}{2} = a+c$
② $\frac{1}{2} = 9a+c$

① − ② : $1 = -8a$ $a = \frac{-1}{8}\,;\ c = \frac{13}{8}$

$\Rightarrow p(x) = (x+1)(-\frac{1}{8}x^2 + \frac{13}{8})$
$= \frac{-1}{8}(x+1)(x^2-13)$