Rational Models: Understanding Practice 4

Goal: Be able to model rational functions in general cases and interpret the model in a meaningful way.

Given the following situations, what are important numbers (constants and variables) that are needed to understand and predict things, and what would we want to predict about the situation?

A. A plane is flying between cities while it is windy.

B. Two heaters are being used to heat a cabin.

C. Two solutions of food colouring are being mixed together.

1. In situation A, I imagined a plane travelling between Vancouver and Calgary (1000 km) and the speed of the plane is 600 km/h if there is no wind. The wind is 100 km/h and the plane flies through the headwind for 400 km and flies through the tailwind on the way back for 500 km. I would want to know how long it takes to make a round trip.

   Build an equation to determine the time it would take for a round trip.

$$T = \frac{1100}{600} + \frac{500}{700} + \frac{400}{500}$$

$$= 3 \text{ hr and } 21 \text{ min}$$

2. Generalize this. The plane's speed is p (km/h), the wind's speed is ω, the distance between the cities is d (km) and the plane flies through the headwind for ∂_H (km) and flies through the tailwind for ∂_T .

   Build an equation for the time it takes to make a round trip

$$T(p, d, \omega, \partial_H, \partial_T) = \frac{2d - \partial_H - \partial_T}{p} + \frac{\partial_H}{p - \omega} + \frac{\partial_T}{p + \omega}$$

General

$$T : \mathbb{R}^5 \longrightarrow \mathbb{R}$$

$p, d > 0$

$\partial_H, \partial_T, \omega \geq 0$

Better

$$T : \mathbb{N}^2 \times \mathbb{Z}^3 \longrightarrow \mathbb{Q}$$

output will be a rational

first 2 inputs are natural

last 3 are non-negative integers

Arguable that the speeds and distances would be reported to the nearest whole #

3. Use the following situation: $p = 600$, $d = 1000$, $\partial_H = 500$ and $\partial_T = 500$ to make an equation for the time t as a function of the wind speed ω.

$$t(w) = T(600, 1000, w, 500, 500)$$

$$= \frac{1000}{600} + \frac{500}{600-w} + \frac{500}{600+w}$$

$t: \mathbb{Z} \to \mathbb{R}$

$w \geqslant 0$

4. What is a reasonable domain for the function?

$w \geqslant 0$, but the plane can't fly if $w > 300$ so

$w \in [0, 300] \cap \mathbb{Z}$ seems reasonable

5. Determine the interval of wind speeds that would allow the plane to make the round trip in under 3 hours and 30 minutes.

$$t(w) = \frac{7}{2} = \frac{10}{6} + 500\left(\frac{1}{600+w} + \frac{1}{600-w}\right)$$

$\Rightarrow$

$$\frac{11}{6} \cdot \frac{1}{500} = \frac{1200}{600^2 - w^2}$$

$$600^2 - w^2 = \frac{1200 \cdot 500 \cdot 6}{11}$$

$w = 180.9$ km/h $\Rightarrow$ if the wind is 180 km/h or less the trip will be under 3.5 hrs

6. In situation B I imagined a heavy-duty heater that could raise the temperature in the room by 1°C in 3 minutes and a less impressive heater that could raise the temperature in the room by 1°C in 20 minutes. I would want to know working together how long would it take to heat the room by 20°C (something reasonable like from −5°C to 15°C)

How long would it take the heaters working together to heat the room by 20°C?

$$\frac{1}{3}\ {}^\circ C/min + \frac{1}{20}\ {}^\circ C/min = 0.38\overline{3}\ {}^\circ C/min$$

$$\Rightarrow T = \frac{temp}{rate} = \frac{20\ {}^\circ C}{0.38\overline{3}\ {}^\circ C/min}$$

$$= 52 \text{ min } 10 \text{ sec}$$

$$\sim 52 \text{ min}$$

7. Generalize this. The strong heater will raise the temperature by 1°C in τ_H minutes and the weak heater will raise the temperature by 1°C in τ_h minutes. We want to raise the room's temperature by T degrees Celsius. Make an equation for the time needed t.

$$t(T, \tau_H, \tau_h) = \frac{T}{\frac{1}{\tau_H} + \frac{1}{\tau_h}} = \frac{T}{\tau_h + \tau_H} \cdot \tau_h \cdot \tau_H$$

general

$t: \mathbb{R}^3 \to \mathbb{R}$

$T \geq 0, \quad \tau_H, \tau_h > 0$

Better

$t: \mathbb{Q}^3 \to \mathbb{Q}$

$T \geq 0, \quad \tau_H, \tau_h > 0$

I can imagine wanting to change the temp. by a 0.5°C and I would expect I could have a heater that heats 1°C in 35 min

8. Use the following situation: $T = 20$, $\tau_h = 20$. Make an equation so your time t is a function of τ_H (the time needed for the heavy-duty heater).

$$t_1(\tau_H) = t(20, \tau_H, 20) = \frac{400\,\tau_H}{\tau_H + 20}$$

$t_1 : \mathbb{Q} \to \mathbb{Q}$

$\tau_H > 0$

9. What would be a reasonable domain for τ_H?

see above map. $\tau_H > 0$ (it can **NOT** be 0)

we probably would not be impressed if $\tau_H > 60$ (it takes more than 1 hour to change 1°C)

$\Rightarrow \tau_H \in (0, 60] \cap \mathbb{Q}$

10. What does τ_H need to be in order for the room to gain 20°C in 30 minutes?

$$t_1(\tau_H) = 30 = \frac{400\,\tau_H}{\tau_H + 20}$$

$$\Rightarrow \quad 30\,\tau_H + 600 = 400\,\tau_H$$

$$\tau_H = 1.62 \text{ min}$$

$\Rightarrow$ it needs to change 1°C in 1 min 37 sec

11. In situation C I imagined I had 100mL of a 25% red solution being mixed with 200mL of a 40% solution. I want to know what the concentration of the mixture is.

    Determine the concentration of the mixture.

$$C = \frac{0.25 \cdot 100 + 0.4 \cdot 200}{100 + 200}$$

$$= 35\%$$

12. Generalize this. You have V_0 litres of a solution of concentration c_0 and add V litres of a solution of concentration c. Determine an equation for the concentration of the mixed solution.

$$C(V, V_0, c, c_0) = \frac{cV + c_0V_0}{V + V_0}$$

Technically the range is between $\min(c, c_0)$ and $\max(c, c_0)$

General $C: \mathbb{R}^4 \to \mathbb{R}$

Better $C: \mathbb{Q}^2 \times [0,1]^2 \to [0,1] \cap \mathbb{Q}$

V can be 0 but $V_0 \neq 0$ since we need to have a solution of something to mix.

$V \geq 0$, $V_0 > 0$

$c, c_0 \in \mathbb{Q}$

If $V_0 = 0$ then we just are changing containers

☆ since c, c_0 are % values the max they can be is 1

$c, c_0 \in [0,1] \cap \mathbb{Q}$

13. If $V_0 = 0.2$ and $c_0 = 0.25$, write the concentration of the mixture as a function of V.

$$\mathcal{C}_c(V) = \mathcal{C}(V, 0.2, c, 0.25)$$

$$\mathcal{C}_c(V) = \frac{cV + 0.05}{V + 0.2}$$

$[0.25, c)$ if $c > 0.25$

$\mathcal{C}_c : \mathbb{Q} \to (0,1) \cap \mathbb{Q}$

$(c, 0.25]$ if $c < 0.25$

subscript to indicate the concentration of solution

14. What is the horizontal asymptote and what does it represent?

$$\mathcal{C}_c(V) \sim \frac{cV}{V} = c \rightarrow$$ The concentration of the solution being added

15. If I want a 30% mixture and have an 80% bottle of red. How much of the 80% do I need to add?

$$\mathcal{C}_{0.8}(V) = \frac{0.8V + 0.05}{V + 0.2} = 0.3$$

$$\Rightarrow \quad 0.8V + 0.05 = 0.3V + 0.06$$

$$V = 0.02$$

$\Rightarrow$ add 20 mL of the 80% solution.