Exponential Models: Understanding Practice 6

Goal: Be able to model exponential functions in general cases and interpret the model in a meaningful way.

Given the following situations, what are important numbers (constants and variables) that are needed to understand and predict things, and what would we want to predict about the situation?

A. An amount of money is invested and left to grow.

B. The population of salmon was stable but has started to decline due to over fishing and environmental stresses in a region.

C. A baked pie comes out of the oven and cools down.

D. A rumour is spread around the school until eventually everyone hears it.

In situation A, I imagined an investment of $\$50{,}000$ is expected to return 7% annually (every year).

Determine a function for the amount the investment grows after t years.

$$A(t) = 50(1+0.07)^t = 50\, e^{t \ln 1.07}$$

Generalize this. The initial investment is A_0 and the rate of return is r. State the mapping notation of this function and describe its domain. Be sure to specify a reasonable domain for t.

$$A(A_0, r, t) = A_0(1+r)^t = A_0 e^{t \ln(1+r)}$$

$$A : \mathbb{N} \times \mathbb{Q}^2 \longrightarrow \mathbb{N}$$

$A, A_0 \in \mathbb{N}$ some natural # for the \$ amount in the account.

$r \in \mathbb{Q} \cap [0,1)$ and $t \geq 0$ since at $t=0$ we just opened the account

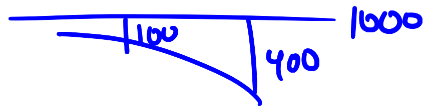
Use the situation: $A_0 = \$80{,}000$ and $r = 9\%$ to determine the time when the investment has grown to $\$500{,}000$.

$$500 = 80(1+0.09)^t$$

$$\ln\left(\frac{50}{8}\right) = t \ln 1.09$$

$$\Rightarrow t = \frac{\ln(50/8)}{\ln(1.09)} = 21.26 \text{ yrs}$$

$\Rightarrow$ 21 years + 4 months



In situation B I imagined that the population was stable at 1 million, but in 1985 the population was 900K and in 1990 the population was 600K.

Determine a function for the salmon population in year t.

$$P(t) = -100\,(4)^{\frac{t-1985}{5}} + 1000$$

$P(t)$ is measured in thousands

Generalize this. The stable population is P_0, and there were P_1 fish in year t_1 and P_2 fish in year t_2. We have that $P_0 > P_1 > P_0$ and $t_2 > t_1$. State the mapping notation of this function and describe its domain. Be sure to specify a reasonable domain for t.

$$P(P_0, P_1, P_2, t_1, t_2, t) = (P_1 - P_0)\left(\frac{P_2 - P_0}{P_1 - P_0}\right)^{\frac{t-t_1}{t_2-t_1}} + P_0$$

$$= (P_1 - P_0)\exp\left(\left(\frac{t-t_1}{t_2-t_1}\right)\ln\left(\frac{P_2-P_0}{P_1-P_0}\right)\right) + P_0$$

$$P: \mathbb{N}^3 \times \mathbb{Q}^3 \to \mathbb{N}$$

$P_k \in \mathbb{N}$ and $t_k \in \mathbb{Q}$ with $t < t_2 + 20$ as the model will break down for large t

Use the original parameters to predict when the population will become 400K.

$$400 = -100\exp\left(\ln 4 \cdot \frac{(t - 1985)}{5}\right) + 1000$$

$$6 = e^{\frac{1}{5}\ln 4\,(t-1985)}$$

$$\frac{5\ln 6}{\ln 4} + 1985 = t = 1991.5$$

$\Rightarrow$ June 1991

In situation C I imagined a baked pie comes out of the oven and is 80°C at 4pm, it is left to cool until dinner at 7pm when the temperature is 24°C. The temperature of the room is 20°C.

Determine a function for the temperature of the pie at a given time.

$$T(t) = 60\left(\frac{4}{60}\right)^{\frac{t-4}{3}} + 20$$

$$= 60\, e^{-\frac{1}{3}\ln 15 \cdot (t-4)} + 20$$

To generalize this, think of how is this situation similar/different to the salmon problem? For the above situation, let the room temperature T_{amb}, and say the pie was T_1 degrees at time t_1 (pm hours) and T_2 degrees at time t_2 ([redacted]). State the mapping notation of this function and describe its domain. Be sure to specify a reasonable domain for t.

$$T(T_{amb}, T_1, T_2, t_1, t_2, t) = (T_1 - T_{amb})\exp\left(\left(\frac{t-t_1}{t_2-t_1}\right)\ln\left(\frac{T_2-T_{amb}}{T_1-T_{amb}}\right)\right) + T_{amb}$$

$$T: \mathbb{Q}^6 \to \mathbb{Q}$$

Temperatures and times having decimals would be good.
$t \geq t_1$ since this is when we started measuring.

What time will the pie become room temperature (use the case at the top)?

let $20.1°C \equiv 20°C$

$$\Rightarrow 20.1 = 60\, e^{\left(\frac{t-4}{3}\right)\ln\left(\frac{4}{60}\right)} + 20$$

$$\Rightarrow \frac{3\ln\frac{0.1}{60}}{\ln(4/60)} + 4 = t = 11.08 \Rightarrow \underline{11:05 \text{ pm}}$$

In situation D I imagined that Prince of Wales has about 1000 students and on June 15th 500 people have heard the rumour and on June 22nd 700 people have herd it.

Determine a function that describes the number of people who have heard the rumour at time t.

$$N(t) = -500\left(\frac{3}{5}\right)^{\frac{t-15}{7}} + 1000$$

$$= -500\, e^{\frac{1}{7}\ln(3/5)(t-15)} + 1000$$

Generalize this using your own variables. State the mapping notation of this function and describe its domain. Be sure to specify a reasonable domain for t.

N students n_1 @ t_1 n_2 @ t_2

$$R(N, n_1, n_2, t_1, t_2, t) = (n_1 - N)\exp\left(\left(\frac{t-t_1}{t_2-t_1}\right)\ln\left(\frac{n_2-N}{n_1-N}\right)\right) + N$$

$$R : \mathbb{N}^6 \to \mathbb{N}$$

$N, n_k \in \mathbb{N}$ and $t_k \in \mathbb{N}$ as I don't want to consider midway thru day. * $t \geq t_1$

Use your original equation to determine when everyone has heard the rumour.

999.5 students $\equiv$ 1000 students . $t=1$ is June 1st

$$\Rightarrow \quad 999.5 = -500\, e^{\ln\left(\frac{3}{5}\right)\cdot\left(\frac{t-15}{7}\right)} + 1000$$

$$\frac{7\ln\left(\frac{0.5}{500}\right)}{\ln(3/5)} + 15 = t = 109.6 = 110$$

$\sim$ Sept $\underline{18^{th}}$

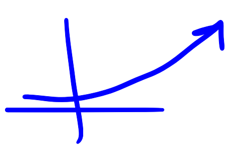
t	date
1	Jun 1
31	July 1
62	Aug 1
93	Sep 1
110	Sep 18

+17 +17

Why would we need a different model if we wanted to model the beginning of the rumour spreading?

At the beginning the rumour spreads very quickly

like:



at the end we have