Exponential and Log Transformations
Understanding Practice 7

Goal: Use exponential and log laws to relate transformations into different forms of each other.

1. Consider the transformation of the exponential function:

$$f(x) = 2 \cdot e^{0.5(x-4)} - 1$$

a. State how f has been transformed from e^x

1.) vert + horiz expand by 2

2.) right 4 and down 1

b. Write this transformation in the form $Ae^{kx} + D$, state the transformations that have occurred.

$2e^{0.5x-2} - 1 = 2e^{-2} \cdot e^{0.5x} - 1$

1.) vert stretch by $\frac{2}{e^2}$ and horiz. expand by 2

2.) down 1

c. Write this transformation in the form $e^{k(x-C)} + D$, state the transformations that have occurred.

$2e^{0.5(x-4)} - 1 = e^{\ln 2} \cdot e^{\frac{1}{2}(x-4)} - 1 = e^{\frac{1}{2}x - 2 + \ln 2} - 1$

$= e^{\frac{1}{2}(x - 4 + 2\ln 2)} - 1$

1.) horiz expand by 2

2.) shift down 1 and right $4 - \ln 4$

OR

1.) right by $2 - \ln 2$

2.) horiz expand by 2

3.) down by 1

2. Consider the transformation of the log function:

$$h(x) = 2\ln\left(\frac{1}{3}(x+3)\right) + 1$$

a. State how h has been transformed from $\ln x$.

1.) vert expand by 2 + horiz expand by 3

2.) left 3 and up 1

b. Write this transformation in the form $A \cdot \ln(x - C) + D$, state the transformations that have occurred.

$2\ln\left(\frac{1}{3}(x+3)\right)+1 = 2\ln\frac{1}{3} + 2\ln(x+3) + 1$

1.) vert expand by 2

2.) left 3 and down $2\ln 3 - 1$

c. Write this transformation in the form $A \cdot \ln\big(B(x - c)\big)$, state the transformations that have occurred.

$2\ln\frac{1}{3}(x+3) + 1 = 2\ln\frac{1}{3}(x+3) + 2\ln e^{\frac{1}{2}}$

$= 2\left[\ln\left(\frac{1}{3}(x+3)\cdot e^{1/2}\right)\right]$

$= 2\ln\left(\frac{e^{1/2}}{3}(x+3)\right)$

1.) vert expand by 2

2.) horiz expand by $\frac{3}{\sqrt{e}}$

3.) left by 3

3. Consider the general transformation of the exponential function:

$$g(x) = a \cdot e^{k(x-c)} + d$$

a. Write g in the form $Ae^{Kx} + D$, where A, K, D are in terms of a, k, c, d. State which transformations are equivalent.

$$a \cdot e^{k(x-c)} + d = ae^{kx} e^{-kc} + d$$

$$A = ae^{-kc}, \quad K = k, \quad D = d$$

$\Rightarrow$ a horizontal shift by c

$\equiv$

a vertical stretch by e^{-kc}

(where $\frac{1}{K}$ is the horizontal stretch)

☆ if $c>0$ right shift $\Rightarrow$ vertical compression

if $c<0$ left shift $\Rightarrow$ vertical expansion

b. Write g in the form $e^{K(x-C)} + D$, where C, K, D are in terms of a, k, c, d. State which transformations are equivalent.

$$ae^{k(x-c)} + d = e^{\ln a} \cdot e^{k(x-c)} + d, \quad a>0$$

(☆ if $a<0$ we can't do this)

$$= e^{kx - kc + \ln a} + d$$

$$= e^{k(x-(c-\frac{1}{k}\ln a))} + d$$

$$\Rightarrow C = c - \frac{1}{k}\ln a, \quad K = k, \quad D = d$$

$\Rightarrow$ a vertical stretch by a

$\equiv$

a horiz. shift by $-\frac{1}{k}\ln a$ ($\frac{1}{k}$ is the horiz. stretch)

☆ if $a \in (0,1) \Rightarrow$ vertical compression $\Rightarrow$ shift right

if $a>1 \Rightarrow$ vertical expansion $\Rightarrow$ shift left

c. Why would the form in part (a) preferred over (b)? Hint: Consider what transformation couldn't occur if there were no vertical stretch.

as shown in the previous, we can't bring a -ve (Rox) into a shift. There would still be a -ve in front.

d. Show that a transformation in part (a) implies the transformation in part (b) and vice versa.

(a) h. shift by $c \equiv$ v. stretch by e^{-kc}

let $-kc = m$

$\Leftrightarrow$ h. shift by $-\frac{m}{k} \equiv$ v. stretch by e^m

let $e^m = A$

$\Leftrightarrow$ h. shift by $-\frac{\ln A}{k} \equiv$ v. stretch by A

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and $\mathcal{F}(x_0) = e^{-kx_0}$ and $\mathcal{G}(y_0) = -\frac{\ln y_0}{k}$

$$\mathcal{F}(\mathcal{G}(x)) = e^{-k\left(-\frac{\ln x}{k}\right)} = e^{\ln x} = x$$

$$\mathcal{G}(\mathcal{F}(x)) = -\frac{\ln(e^{-kx})}{k} = \frac{kx}{k} = x \quad \checkmark$$

4. Consider the general transformation of the log function:

$$k(x) = a \cdot \ln\big(b(x-c)\big) + d$$

a. Write k in the form $A \cdot \ln(x - C) + D$, where A, C, D are in terms of a, b, c, d. State which transformations are equivalent.

$a \ln b(x-c) + d = a\ln b + a\ln(x-c) + d$ ☆ $b>0$

$A = a$; $C = c$; $D = a\ln b + d$

$\Rightarrow$ a horiz. stretch by $\frac{1}{b}$

$\equiv$

vert. shift by $a\ln b$ (where a is vert. stretch)

☆ if $b \in (0,1) \Rightarrow$ h. expansion $\Rightarrow$ shift down $(a>0)$

if $b > 1 \Rightarrow$ h. compression $\Rightarrow$ shift up $(a>0)$

b. Write k in the form $A \cdot \ln\big(B(x - C)\big)$, where A, B, C are in terms of a, b, c, d. State which transformations are equivalent.

$a \ln b(x-c) + d = a\ln b(x-c) + \ln e^d$

$= a\left(\ln b(x-c) + \frac{1}{a}\ln e^d\right)$

$= a \ln\left(be^{\frac{d}{a}}(x-c)\right)$

$\Rightarrow A = a$; $B = be^{\frac{d}{a}}$; $C = c$

so a vert. shift by d

$\equiv$

a horiz stretch by $e^{-\frac{d}{a}}$ (a is vert. stretch)

☆ $d>0 \Rightarrow$ shift up $\Rightarrow$ horiz compress $(a>0)$

$d<0 \Rightarrow$ shift down $\Rightarrow$ horiz. expansion $(a<0)$

c. Why would the form in part (b) preferred over (a)?

As seen before part (b) allows to have ROY since part (a) we need to evaluate $\ln b$, $b>0$

e. Show that a transformation in part (a) implies the transformation in part (b) and vice versa.

(a) h. stretch by $\frac{1}{b}$ $\equiv$ v. shift $a \ln b$

let $a \ln b = D \iff \frac{D}{a} = \ln b \iff e^{\frac{D}{a}} = b$

$\iff$ h. stretch by $\frac{1}{e^{D/a}}$ $\equiv$ v. shift D

h. stretch by $e^{-D/a} \equiv$ v. shift D

(b)

$\mathcal{F}(y_0) = a \ln \frac{1}{y_0}$ $\quad$ $\mathcal{G}(x_0) = e^{-\frac{x_0}{a}}$

$\mathcal{F}(\mathcal{G}(x)) = a \ln \frac{1}{e^{-\frac{x}{a}}} = a \ln e^{\frac{x}{a}} = \frac{ax}{a} = x$

$\mathcal{G}(\mathcal{F}(x)) = e^{-\frac{a \ln \frac{1}{x}}{a}} = e^{-\ln \frac{1}{x}} = e^{\ln x} = x$ ✓