Applied Rates of Change – More Practice

Goal:
• Understands how to relate variables to time and to each other when taking differential equations • Can use geometric equations and understands their applications • Can use scientific equations involving a differential equation
Terminology: • Differential Equation

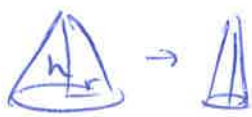
Volume: The volume V of a cone is related to the radius r and height h and can be expressed as

$$V = \frac{1}{3}\pi r^2 h$$

a. How is dV/dt related to dr/dt if h is constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt}$$

⇒ The radius is changing but height is constant so a cone is being rolled up



b. How does the change in volume with respect to time relate to the change in radius and height if neither are constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

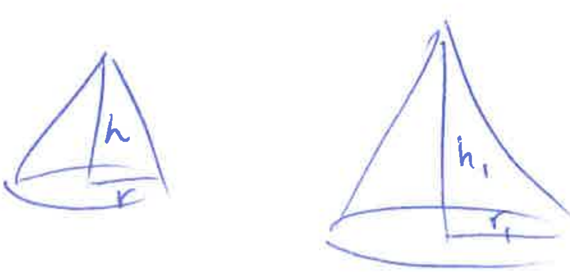
$$\frac{dV}{dt} = \frac{1}{3}\pi\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$

⇒ The cone is growing/shrinking not neccessarily proportional

c. How is the change in volume with respect to height related to the change in radius if neither r or h are constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$$\frac{dV}{dh} = \frac{1}{3}\pi\left[r^2 + 2rh\frac{dr}{dh}\right]$$

⇒ measure the change in volume to how the height changes. Still grows not neecessarily proportional.



know $\frac{dr}{dh}$ get $\frac{dV}{dh}$

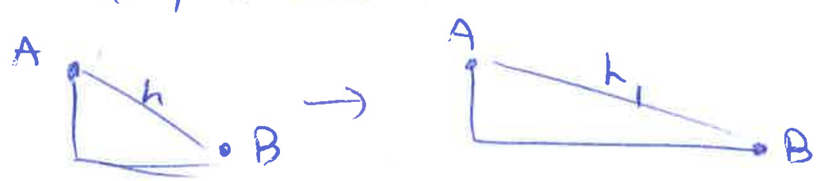
Distance: Let x and y be the horizontal and vertical distance between two points. Then the distance between then points is

$$h = \sqrt{x^2 + y^2}$$

a. How is dh/dt related to dx/dt if y is constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$$\frac{dh}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt}\right)$$

→ moving East/West not North/South

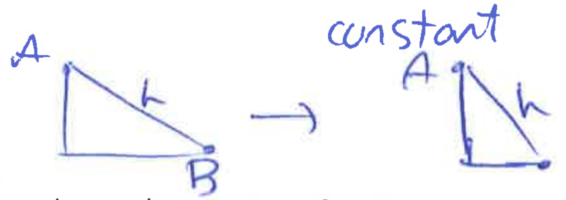


b. How is the change in x with respect to time related to the change in y if h is constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$$0 = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}\left(\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}\right)$$

→ points move but overall distance is constant

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dx}$$



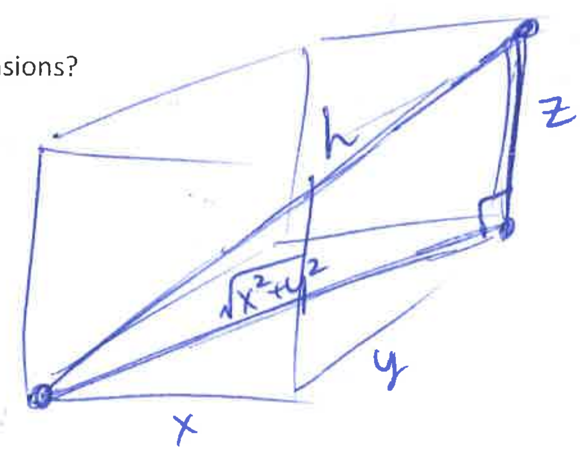
c. How is the change in h related to the change in x and y over time if neither are constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$$\frac{dh}{dt} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

→ points are moving freely.

d. How would this equation be extended for 3 dimensions?

$$h = \sqrt{x^2+y^2+z^2}$$

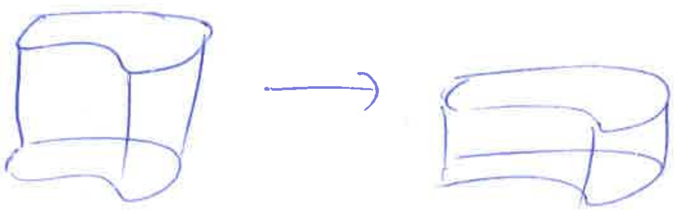


Volume of Prism: The volume of a prism is related to the area of the base A and the height h as

$$V = A \cdot h$$

a. How is dV/dt related to dh/dt if A is constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$\frac{dV}{dt} = A\frac{dh}{dt}$ → The prism is growing taller or shorter (squished or stretched)



same area of base

b. How is the change ~~in volume related to the change~~ in height ~~and~~ related to change in area over time if ~~neither are constant?~~ volume is const. Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$0 = A\frac{dh}{dt} + h\frac{dA}{dt}$ $\quad \frac{dh}{dt} = \frac{-h}{A}\frac{dA}{dt}$ → constant volume just change dimensions (like play dough or clay)

same volume

c. How is $\boxed{dA/dh}$ related to dV/dt if neither are constant? ~~Write a sentence to describe a scenario where this would occur and draw a picture that models it.~~

$\cancel{\frac{dV}{dt} = A\frac{dh}{dt}}$ $\quad \frac{dA}{dh} = ?$ $\quad \frac{dV}{dh} = h\frac{dA}{dh} + A$

$\frac{dV}{dt} = h\frac{dA}{dt} + A\frac{dh}{dt}$ $\quad \cancel{\frac{dA}{dh} = \frac{dA}{dt} \Big/ \frac{dt}{dh} \Big/ \frac{dA}{dt} \Big/ \frac{A}{dV/dt}}$

chain rule: $\frac{dA}{dh} = \frac{dA}{dt}\frac{dt}{dh}$

$\Rightarrow \frac{dV}{dt}\cdot\frac{dt}{dh} = h\boxed{\frac{dA}{dt}\cdot\frac{dt}{dh}} + A\frac{dh}{dt}\cdot\frac{dt}{dh} \Rightarrow$

$$\frac{dA}{dh} = \frac{\frac{dV}{dt}\frac{dt}{dh} - A}{h}$$

Economics: The cost to produce n units is $C(n)$ and the revenue is $R(n)$.

a. Determine an expression for the profit to produce n units, $P(n)$

$$P(n) = R(n) - C(n)$$

b. How does dP/dn relate to dC/dn and dR/dn? Write a sentence to describe a scenario where dC/dn is constant but dR/dn is variable.

$$\frac{dP}{dn} = \frac{dR}{dn} - \frac{dC}{dn}$$

If $\frac{dC}{dn}$ is const then it costs the same to prod. a unit always, but $\frac{dR}{dn}$ changes b/c if we sell in bulk we make less revenue.

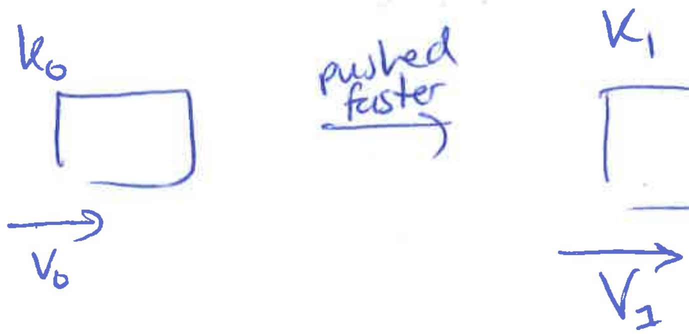
Physics: The kinetic energy of a moving object is related to its velocity and mass

$$K = \frac{1}{2}mv^2$$

a. How does dK/dt relate to dv/dt if mass is constant? Write a sentence to describe a scenario where this would occur and draw a picture that models it.

$$\frac{dK}{dt} = mv\frac{dV}{dt}$$

change in Kinetic energy is proportional to its change in velocity.



b. The force is related to mass and acceleration of an object

$$F = ma$$

How does F relate to the change in kinetic energy with respect to time?

$$\frac{dk}{dt} = mv\frac{dv}{dt} \qquad a = \frac{dv}{dt} = \frac{F}{m}$$

$$\boxed{\frac{dk}{dt} = Fv}$$

c. Work is related to force and distance

$$W = Fd$$

How does W relate to the change in kinetic energy with respect to time? Interpret the results by considering $\Delta K/\Delta t$

$$F = \frac{W}{d} \Rightarrow \frac{dk}{dt} = F \cdot v = W\frac{v}{d}$$

$$v = \frac{d}{t} \quad \text{so} \quad \frac{v}{d} = \frac{1}{t}$$

$$\frac{\Delta k}{\Delta t} = W \cdot \frac{\Delta v}{\Delta t} = \frac{W}{\Delta t} \Rightarrow \Delta k = W$$

change in kinetic energy is work!

Chemistry: For an ideal gas pressure, volume and temperature are related by

$$PV = nRT$$

Where n and R are constants.

a. How does the volume change with respect to temperature if pressure is constant?

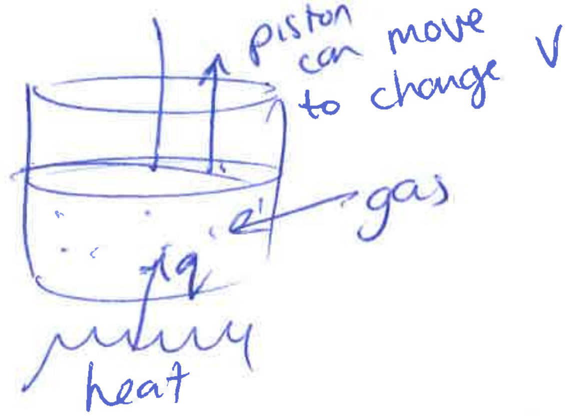
$$P\frac{dV}{dT} = nR \Rightarrow \frac{dV}{dT} = \frac{nR}{P}$$

b. For an ideal gas, the kinetic energy of a closed system is related to the heat added q and the energy lost by the gas changing volume.

$$E = q - PV$$

Relate the change in kinetic energy with respect to the change in volume. Draw a picture to illustrate heating a closed container of helium, such that the volume can change.

$$\frac{dE}{dV} = \frac{dq}{dV} - P$$



c. For monotonic ideal gasses (like helium), the kinetic energy is related to the change in its temperature as

$$E = \frac{3}{2}nRT$$

Relate the change in kinetic energy with respect to the change in the temperature of the gas.

$$\frac{dE}{dT} = \frac{3}{2}nR$$

d. The specific heat capacity of a gas is how much heat (Δq) is needed to change the temperature of the gas (ΔT). Show that the specific heat capacity at constant pressure is

$$\frac{dq}{dT} = \frac{5}{2}nR$$

$$E = \frac{3}{2}nRT$$

$$E = \frac{3}{2}PV$$

$$\frac{dE}{dV} = \frac{3}{2}P$$

$$\frac{dq}{dT} = \frac{dq}{dV} \cdot \frac{dV}{dT}$$

$$= \left(\frac{dE}{dV} + P\right)\left(\frac{nR}{P}\right)$$

$$= \left(\frac{5}{2}P\right)(nR/P) = \frac{5}{2}nR \quad @ \text{ const. pressure.}$$

e. Recall that the kinetic energy of a gas only depends on change in heat added and change in volume (not the change in pressure). Deduce that the specific heat capacity at constant volume is $\frac{3}{2}nR$

$$E = q - PV \qquad \text{const } V.$$

$$\frac{dE}{dT} = \frac{dq}{dT} - 0 \leftarrow \text{const.}$$

$$\boxed{\frac{dq}{dT} = \frac{3}{2}nR}$$

const. volume.