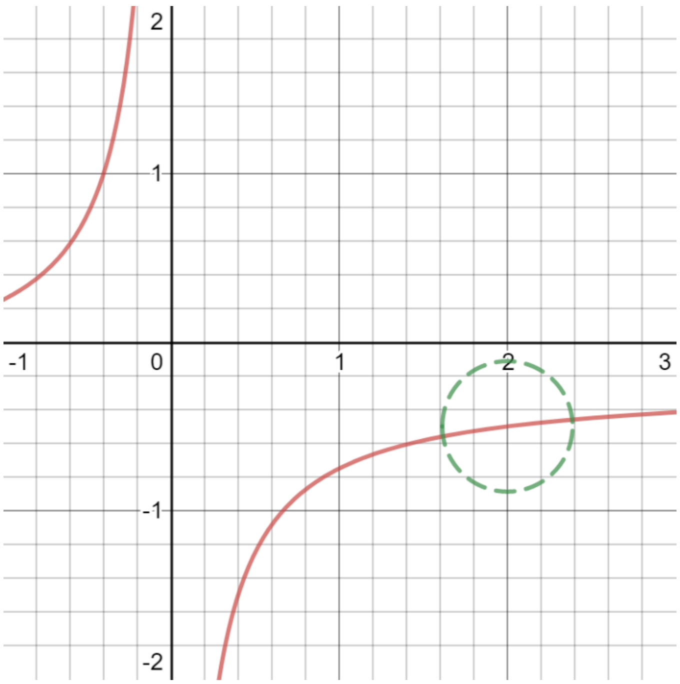
The Limit

We looked at finding the limit as the value the neighbourhood of a function approaches for arbitrarily small neighbuorhoods around a point $x = c$. Consider

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{x}{4}}{x-2}$$



Then the graph reveals that as we tighten up our neighbourhood around $x = 2$ that $f(x)$ looks like it approaches a value around -0.5.

A table of values gives us a better picture and we see that the limit is perhaps exactly -0.5 (at least to four decimal places).

The only way to do better is to use algebra and simplify

$$\begin{aligned}\lim_{x\to 2}\frac{\frac{1}{x}-\frac{x}{4}}{x-2} &= \lim_{x\to 2}\frac{4-x^2}{4x(x-2)}\\ &= \lim_{x\to 2}\frac{(2-x)(2+x)}{4x(x-2)}\\ &= \lim_{x\to 2}\frac{-(2+x)}{4x}\\ &= -\frac{4}{8} = -\frac{1}{2}\end{aligned}$$

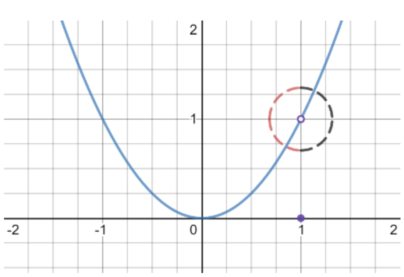
Which is exactly what we thought

x	1.99	1.999	1.9999	2.0001	2.001	2.01
$f(x)$	-0.50126	-0.50013	-0.50001	-0.49999	-0.49988	-0.49876

We say the limit does not exist if it does not approach a real number from both sides (left and right) and in the case above where there is an asymptote at $x = 0$ we could say that $\lim_{x\to 0^-}\frac{\frac{1}{x}-\frac{x}{4}}{x-2} = \infty$ and $\lim_{x\to 0^+}\frac{\frac{1}{x}-\frac{x}{4}}{x-2} = -\infty$ BUT we prefer to say the limit does not exist as infinity in not a number and we can not encircle it like we can real limit points.

For one sided limit, we used the idea of a half ball around the point and that the full limit only exists if both sides are the same. This led us to describe continuity as f is continuous at $x = c$ if and only if $\lim_{x\to c} f(x) = f(c)$

Hole



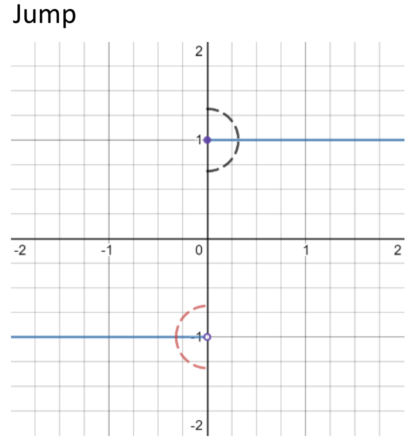
$$f(x) = \begin{cases} x^2, x \neq 1 \\ 0, x = 1 \end{cases}$$

Here $\lim_{x\to 1^-} f(x) = 1$
And $\lim_{x\to 1^+} f(x) = 1$

The limit at 1 is 1, but $f(1) = 0$

Not continuous at 1, but it could be made continuous if we changed f so that $f(0) = 1$.

Jump



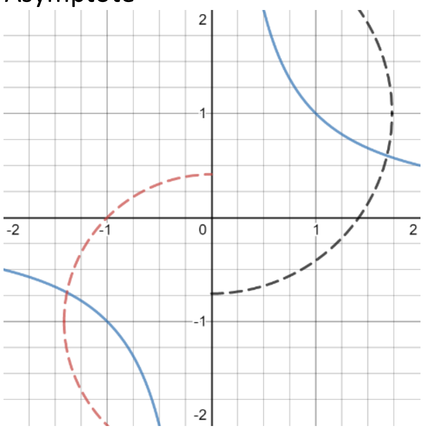
$$f(x) = \begin{cases} -1, x < 0 \\ 1, x \geq 0 \end{cases}$$

Here $\lim_{x\to 0^-} f(x) = -1$
And $\lim_{x\to 0^+} f(x) = 1$

The limit at 0 does not exist, but $f(0) = 1$

Not continuous at 0

Asymptote



$$f(x) = \frac{1}{x}$$

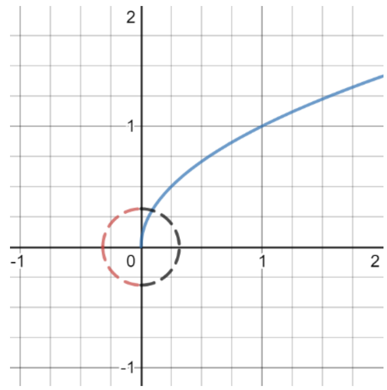
$\lim_{x \to 0^-} f(x) =$ DNE

$\lim_{x \to 0^+} f(x) =$ DNE

The limit at 0 DNE, and $f(0)$ is undefined

Not continuous at 0

Endpoint



$$f(x) = \sqrt{x}$$

$\lim_{x \to 0^-} f(x) =$ undefined

$\lim_{x \to 0^+} f(x) = 0$

The limit at 0 is 0 since we can only talk about values of x in the domain of f. And since $f(0) = 0$ we do have continuous at 0.

Things we need to know and understand:

- How to calculate the limit from a graph or table of values
- How to calculate the limit using algebra (factor and conjugate)
- That the limit exists when its one-sided limits approach the same value.
- How to determine continuity at a point
- How to fill in a removeable discontinuity

Review Questions:

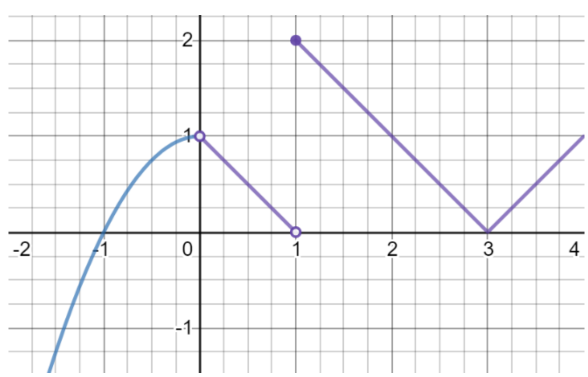
1. What is a limit?
2. Why does $\lim_{x \to 0} \frac{|x|}{x}$ not exist?
3. Graph the function $f(x) = \begin{cases} 1 - x^2, & x < 0 \\ 1 - x, & 0 < x < 1 \\ |x - 3|, & x \geq 1 \end{cases}$ and determine where it is discontinuous. If possible, fill in any removeable discontinuities.
4. Determine $\lim_{x \to 1} \frac{\frac{3}{\sqrt{x+8}} - \sqrt{x}}{x - 1}$.
5. Show that $f(x) = \begin{cases} \frac{x-2}{\sqrt{x+2} - 2}, & x < 2 \\ 4, & x = 2 \\ \frac{8 - 4x}{\frac{1}{x-1} - 1}, & x > 2 \end{cases}$ is continuous at $x = 2$.

Solutions:

1. A value (real number), L, that we can make arbitrarily close to the value of the function if we pick values of x close enough to the limit point.
2. Because $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$ and $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$ which are not the same.
3. It is discontinuous at $x = 0$ and $x = 1$ for different reasons. First $f(0)$ is undefined and second the limit as $x \to 1$ does not exist.

   We could fill in the discontinuity at $x = 0$ by adding

$$f(x) = \begin{cases} 1 - x^2, & x < 0 \\ 1, & x = 0 \\ 1 - x, & 0 < x < 1 \\ |x - 3|, & x \geq 1 \end{cases}$$



4. $\lim_{x\to 1}\frac{\frac{3}{\sqrt{x+8}}-\sqrt{x}}{x-1} = \lim_{x\to 1}\frac{3-\sqrt{x+8}\cdot\sqrt{x}}{(x-1)(2-x)}\cdot\frac{3+\sqrt{x+8}\cdot\sqrt{x}}{3+\sqrt{x+8}\cdot\sqrt{x}} = \lim_{x\to 1}\frac{9-(x+8)(x)}{(x-1)(2-x)(3+\sqrt{x+8}\cdot\sqrt{x})} = \lim_{x\to 1}\frac{-x^2-8x+9}{(x-1)(2-x)(3+\sqrt{x+8}\cdot\sqrt{x})} =$
$\lim_{x\to 1}\frac{-(x+9)(x-1)}{(x-1)(2-x)(3+\sqrt{x+8}\cdot\sqrt{x})} = \lim_{x\to 1}\frac{-(x+9)}{(2-x)(3+\sqrt{x+8}\cdot\sqrt{x})} = -\frac{10}{6} = -\frac{5}{3}$
5. Both the one-sided limits and $f(2) = 4$. We have $\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-2} = \lim_{x\to 2}\frac{(x-2)(\sqrt{x+2}+2)}{x-2} = 4$ and
$\lim_{x\to 2}\frac{8-4x}{\frac{1}{x-1}-1} = \lim_{x\to 2}\frac{(8-4x)(x-1)}{1-(x-1)} = 4\cdot\lim_{x\to 2}\frac{(2-x)(x-1)}{2-x} = 4$

Rates of Change

We looked at the slope of curves at a point of tangency and saw that the slope at $x = c$ was a limit, namely

$$\text{Tangent slope} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{x\to x_0}\frac{y-y_0}{x-x_0} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

This slope is a rate of change and when take the limit we are measuring the instantaneous rate of change at a point. We first found this slope using secant lines with points that were getting closer together which we could calculate with a table (**Average Rate of Change**) and then we used the idea of a limit which we could solve (hopefully) algebraically (**Instantaneous Rate of Change**).

The important idea is that these rates of change have units and those units give content to our problems. The slope is not just a number but a number that represents how fast the y values are changing as the x values change.

Things we need to know and understand:

- How to find the slope at a point using limits and a table.
- How to find the slope of a curve at a general point $x = c$.
- How to find the average rate of change over some interval.

Review Questions:

6. Determine the slope of $y = x^3 - x$ at the point $x = c$ and check your solution by letting $c = 1$ and finding the slope at $x = 1$ using secant lines.
7. Determine the slope of $y = \frac{x+1}{x-1}$ at the point $x = c$ and check your solution by letting $c = 0$ and finding the slope at $x = 0$ using secant lines.
8. Determine the equation of the tangent line to $y = \sqrt{x^2 + x}$ at the point $x = c$ and check your solution by letting $c = 2$ and finding the tangent line at $x = 2$ using secant lines.
9. Given the data for the average net worth of single Canadians at different ages determine the average rate of change of net worth on the interval $[27, 60]$, $[27, 50]$, and $[27, 40]$. Approximate the instantaneous rate of change at 27.

Age	27	40	50	60	75
Net Worth ($1000)	97	124	233	478	830

Source:
https://www150.statcan.gc.ca/t1/tbl1/en/tv.action?pid=1110001601&pickMembers%5B0%5D=1.1&pickMembers%5B1%5D=3.7&pickMembers%5B2%5D=5.1&pickMembers%5B3%5D=4.3

10. The volume of water (in kL) in a tank at time t (hours) can be modelled by $V(t) = t(t-2)^2$. Determine the instantaneous rate of change of water in the tank at 60 minutes. Interpret the result.

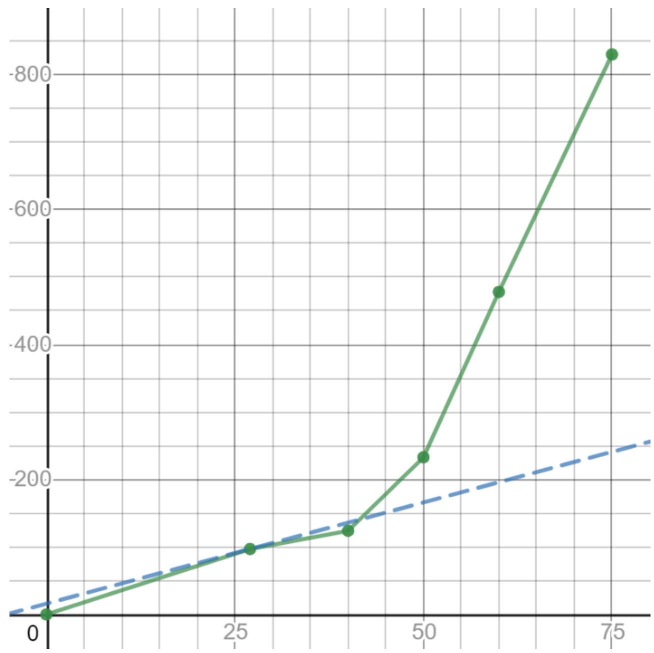
Solutions:

6. $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0}\frac{(c+h)^3-(c+h)-c^3+c}{h}=\lim_{h\to 0}\frac{c^3+3c^2h+3ch^2+h^3-c-h-c^3+c}{h}=3c^2-1$. The slope of the tangent at $x=1$ should be 2.
7. $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{\frac{x+1}{x-1}-\frac{c+1}{c-1}}{x-c}=\lim_{x\to c}\frac{(x+1)(c-1)-(x-1)(c+1)}{(x-1)(c-1)(x-c)}=\lim_{x\to c}\frac{xc-x+c-1-xc-x+c+1}{(x-1)(c-1)(x-c)}=-\frac{2}{(c-1)^2}$. The slope of the tangent at $x=0$ should be -2.
8. $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{\sqrt{x^2+x}-\sqrt{c^2+c}}{x-c}=\lim_{x\to c}\frac{x^2+x-c^2-c}{(x-c)(\sqrt{x^2+x}+\sqrt{c^2+c})}=\lim_{x\to c}\frac{(x-c)(x+c)+(x-c)}{(x-c)(\sqrt{x^2+x}+\sqrt{c^2+c})}=\frac{2c+1}{2\sqrt{c^2+c}}$ is the slope at $x=c$, so the tangent line will have the form:
$$y-y_0=m(x-x_0)$$
Where $x_0=c$ and $y_0=\sqrt{c^2+c}$
$$y=\frac{2c+1}{2\sqrt{c^2+c}}(x-c)+\sqrt{c^2+c}$$
At the point $x=2$ the tangent line will be
$$y=\frac{5}{2\sqrt{6}}(x-2)+\sqrt{6}$$

Video of the tangent line for variable c: https://www.desmos.com/calculator/l86avfcjcg
9. Data looks like the image on the right.
Average rates of change:

Interval	$\frac{\Delta y}{\Delta x}$
[27,60]	$\frac{478-97}{60-27}=11.5$ thousand\$/year
[27, 50]	$\frac{233-97}{50-27}=5.9$ thousand\$/year
[27, 40]	$\frac{124-97}{40-27}=2.1$ thousand\$/year

To find the instantaneous rate of change you should argue why you are doing what you are doing. It is hard to find data on the net worth of teens so at 18 I would say 10K is fair, but I am going to say at 0 the net worth is 0K. Then on [0,27] the rate of change is 3.6 K\$/year and a weighted average is my instantaneous rate of change $\frac{27}{45}\cdot 3.6+\frac{18}{45}\cdot 2.1=3$ thousand\$/year. This line is the dotted line shown.



10. $\lim_{t\to 1}\frac{V(t)-V(1)}{t-1}=\lim_{t\to 1}\frac{t(t-2)^2-1}{t-1}=\lim_{t\to 1}\frac{t^3-4t^2+4t-1}{t-1}=\lim_{t\to 1}\frac{(t-1)(t^2-3t+1)}{t-1}=-1$kL/h. Water is leaving the tank at a rate of 1000 litres per hour.