Differentiability

A function is said to be **differentiable** if the slope (the derivative) exists at $x = c$. That means that the following limit exists:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

This is the definition of the slope at $x = c$ and so you can use the graph by imagining a tangent line and when it doesn't exist. Instances when it doesn't exist is when there is no well-defined tangent line, or the slope of the tangent line is undefined. See our notes for complete list of cases where differentiability breaks down.

To define the derivative at any point x we need to change our label of $c = x$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{c \to x} \frac{f(c) - f(x)}{c - x}$$

And we denote this as $f'(x)$ or $\frac{d}{dx} f(x)$ or $\frac{df}{dx}$. Since this is slope, we are measuring a rate of change and the $\frac{d}{dx}$ notation reflects this as it means the change of some function with respect to how much x changes. That is for a small change in x how much greater or less will the function change. If we want to know the derivative at point, we will denote it as $f'(c)$ or $\frac{d}{dx} f(x)\Big|_{x=c}$

Things we need to know and understand:

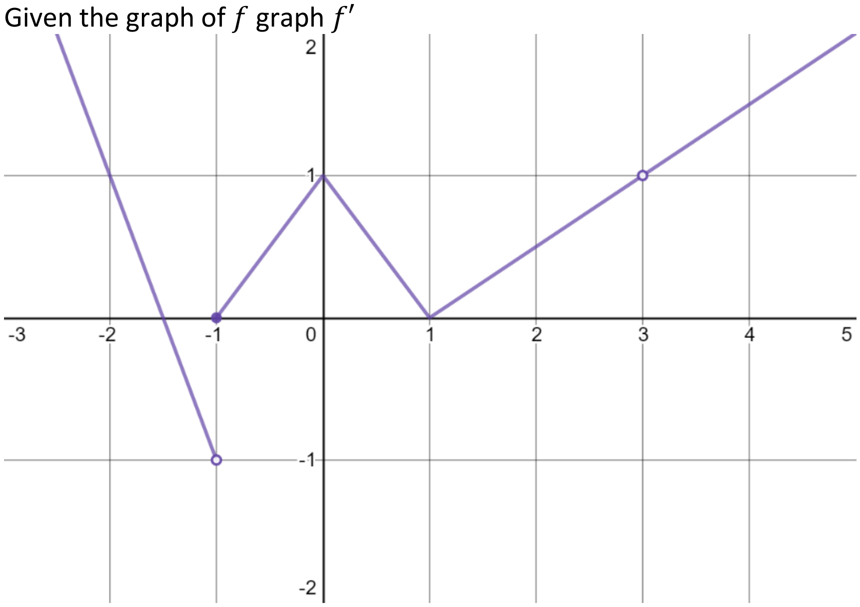
- How to find the slope at a point using limits
- How to find the graph of f' given the graph of f
- How to determine differentiability by comparing the derivative on the left and right of a point

Review Questions:

1. Determine y' using the definition of derivatives if $y = x^2 + x$.
2. Determine a value of m to make the following function differentiable at $x = 1$

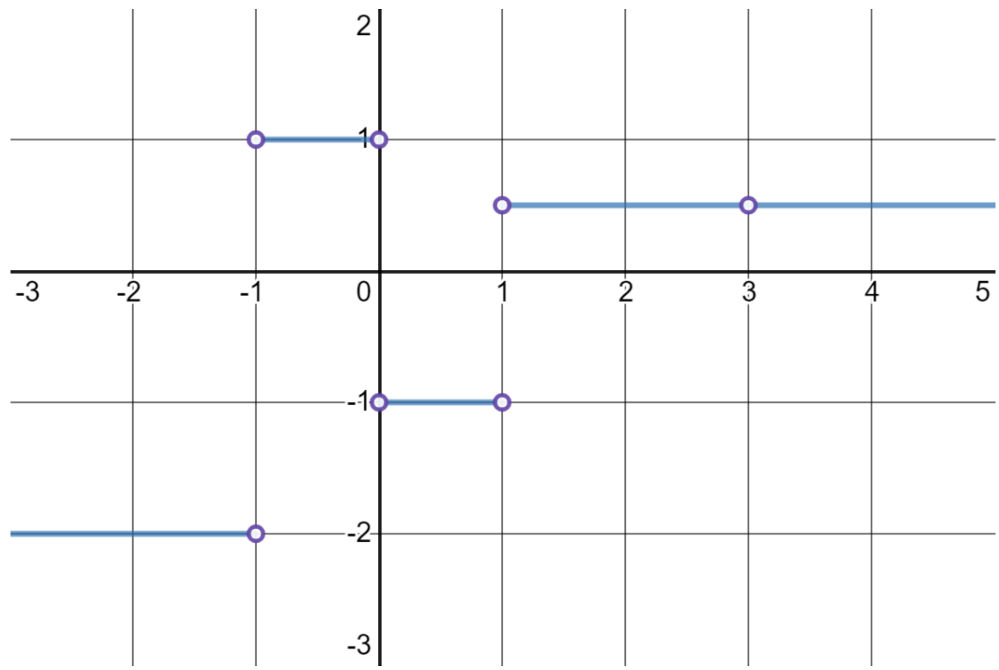
$$y = \begin{cases} \dfrac{x^2}{x+1}, & x \le 1 \\ m(x-1) + \dfrac{1}{2}, & x > 1 \end{cases}$$

3. Given the graph of f graph f'



Solutions:

1. $y' = \lim_{h\to 0} \frac{(x+h)^2+(x+h)-x^2-x}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2+x+h-x^2-x}{h} = \lim_{h\to 0} \frac{2xh+h^2+h}{h} = \lim_{h\to 0}(2x+1+h) = 2x+1$
2. We want the slope on both sides to be the same. Using quotient rule on the left hand side of $x = 1$ the derivative is $y' = \frac{2x(x+1)-x^2}{(x+1)^2}$ and then $y'(1) = \frac{3}{4}$ which needs to be the slope on the right hand side so $m = \frac{3}{4}$
3.



The Basics: Power, Sum, Product, Quotient, Chain

We need to know how to derive these rules and how to use them in context. All of these rules will satisfy the definition of the derivative, but this is not always the best way to illustrate why they are true. The first two limits work nicely and the other 3 we need a visual representation to help. The limit definition will use the expression below:

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

To visualize product and quotient rule using an area model is quite powerful and chain rule works by considering how rates multiply together and units cancel and then takings a limit as the rate goes to 0/0. I leave the method of showing why these rules are true to you below, but the five rules are:

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx}\big(f(x)+g(x)\big) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$\frac{d}{dx}(f \cdot g) = g \cdot \frac{df}{dx} + f \cdot \frac{dg}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$$

$$\frac{d}{dx} f(u) = \frac{df}{du} \cdot \frac{du}{dx}$$

Remember that derivative is slope (rate of change) so these derivates will be used to measure rates of change.

Example: Find $\frac{dy}{dz}$ of the following:

$$y = \sqrt{\frac{xw}{z}}$$

Solution: You should be able to describe the plan of attack before you begin as chain rule; then quotient rule which will use product rule and chain rule inside it giving us dx/dz and dw/dz.

$$\frac{dy}{dz} = \frac{1}{2}\sqrt{\frac{z}{xw}} \cdot \frac{\left(w \cdot \frac{dx}{dz} + x \cdot \frac{dw}{dz}\right)z - xw}{z^2}$$

I would not bother to simplify this.

Things we need to know and understand:

- How to justify the five basic derivative rules
- How to take the derivative of any polynomial function

Review Questions:

4. Prove Power Rule
5. Prove Sum Rule
6. Illustrate Product Rule
7. Illustrate Quotient Rule
8. Justify Chain Rule
9. Find dy/dx of $y = \sqrt[4]{\frac{2x(x+1)}{5x^3-4}}$

Solutions:

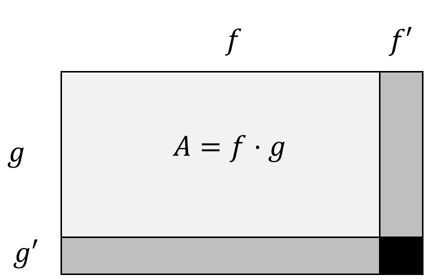
4. Use limits

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n + nx^{n-1}h + \cdots + h^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{nx^{n-1}h + \cdots + h^n}{h} = \lim_{h\to 0} nx^{n-1} + \cdots + h^{n-1} = nx^{n-1}$$

5. Use limits

$$\begin{aligned}\frac{d}{dx}(f+g) &= \lim_{h\to 0}\frac{\big(f(x+h) + g(x+h)\big) - \big(f(x)g + (x)\big)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\\ &= \frac{dg}{dx} + \frac{df}{dx}\end{aligned}$$

6. Use an area model

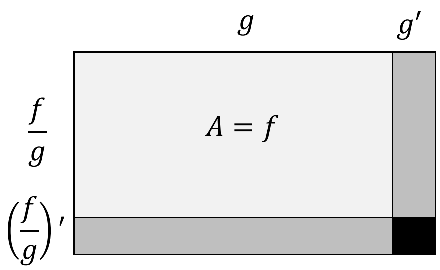


$$A' = (f \cdot g)' = f'g + g'f + g'f'$$

But $g'f'$ is soooo small and is essentially a point once the change in f and g go to 0 so it is treated as 0.

$$(f \cdot g)' = f'g + g'f$$

7. Use the same area model but different labels



$$A' = f' = g' \cdot \frac{f}{g} + \left(\frac{f}{g}\right)' g$$

And we need to solve for $(f/g)'$

$$g^2 \cdot \left(\frac{f}{g}\right)' = f'g - g'f$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

8. We know that

$$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

In the discrete and large world. We also know

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$$

So since $\Delta x \to 0$ we can change our Δ to d

$$\lim_{\substack{\Delta x \to 0 \\ \Delta u \to 0}} \frac{\Delta f}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \frac{df}{du} \cdot \frac{du}{dx}$$

9.

$$\frac{dy}{dx} = \frac{1}{4} \cdot \left(\frac{2x(x+1)}{5x^3 - 4}\right)^{-\frac{3}{4}} \cdot \frac{(4x+2)(5x^3-4) - (15x^2)(2x^2+2x)}{(5x^3-4)^2}$$

Derivative Extra: Implicit and Higher Order

We can use chain rule to take the derivative of any function by taking the derivative to both sides. All we need to remember is chain rule! Always take the derivative of the inside, even if the inside is just a single variable.

Example. Find dy/dx given

$$xy^2 + y = x^3$$

Solution: Take d/dx of both sides

$$\frac{d}{dx}(xy^2 + y) = \frac{d}{dx}(x^3)$$
$$\frac{d}{dx}(xy^2) + \frac{d}{dx}(y) = \frac{d}{dx}(x^3)$$
$$\frac{d}{dx}(x) \cdot y^2 + \frac{d}{dx}(y^2) \cdot x + \frac{dy}{dx} = 3x^2 \cdot \frac{dx}{dx}$$
$$\frac{dx}{dx} \cdot y^2 + 2y \cdot \frac{dy}{dx} \cdot x + \frac{dy}{dx} = 3x^2 \cdot \frac{dx}{dx}$$

Notice that I am taking the derivative of the inside (y and x). I know that the rate of change of x with respect to x is 1, but I do not know what the rate of change of y with respect to x is. This is what we want to solve for!

$$1 \cdot y^2 + 2y \cdot \frac{dy}{dx} \cdot x + \frac{dy}{dx} = 3x^2 \cdot 1$$
$$\frac{dy}{dx} = \frac{3x^2 - y^2}{2xy + 1}$$

This is true for taking the derivative of anything.

Example: Find dy/dt given

$$y + z^2 - \frac{x}{y} = 2$$

Solution: Take d/dt of both sides

$$\frac{d}{dt}\left(y + z^2 - \frac{x}{y}\right) = \frac{d}{dt}2$$

$$\frac{dy}{dt} + 2z \cdot \frac{dz}{dt} - \frac{\frac{dx}{dt} \cdot y - \frac{dy}{dt} \cdot x}{y^2} = 0$$

Let $y' = dy/dt$

$$y' + 2z \cdot z' - x' \cdot \frac{1}{y} + y' \cdot \frac{x}{y^2} = 0$$

$$y' = \frac{\frac{x'}{y} - 2z \cdot z'}{1 + \frac{x}{y^2}}$$

Higher order derivatives are just taking derivative multiple times. Use prime notation up to the third derivative and then use the notation $y^{(n)}$ or $d^n y/dx^n$.

All the rules are the same, but you need to practice taking derivative of chain rule multiple times.

Example: Find d^2y/dx^2 if $y = f(u)$

$$\frac{d}{dx}(y) = \frac{d}{dx}\big(f(u)\big)$$

Let $dy/dx = y'$

$$y' = f'(u) \cdot u'$$

$$\frac{d}{dx}(y') = \frac{d}{dx}(f'(u) \cdot u')$$

$$y'' = \frac{d}{dx}\big(f'(u)\big) \cdot u' + \frac{d}{dx}(u') \cdot f'(u)$$

$$y'' = f''(u) \cdot u' \cdot u' + u'' \cdot f'(u)$$

Things we need to know and understand:

- How to take the derivative of a relation of many variables using implicit differentiation
- How to find higher order derivatives
- How to take higher order derivatives of chain rule

Review Questions:

10. Find dy/dx if $x^3y^2 + 3x = 4y$
11. Find d^2y/dx^2 if $xy = x^2 - y^3$
12. Find $y^{(n)}$ if $y = \frac{1}{x}$
13. Find d^3y/dx^3 if $y = f(g \cdot h)$

Solutions:

10.

$$3x^2 \cdot y^2 + 2y \cdot \frac{dy}{dx} \cdot x^3 + 3 = 4 \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{3x^2y^2 + 3}{4 - 2yx^3}$$

11.

$$y + x \cdot \frac{dy}{dx} = 2x - 3y^2 \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x - y}{x + 3y^2}$$
$$\frac{d^2y}{dx^2} = \frac{\left(2 - \frac{dy}{dx}\right)(x + 3y^2) - \left(1 + 6y \cdot \frac{dy}{dx}\right)(2x - y)}{(x + 3y^2)^2}$$

12.

$$y' = -\frac{1}{x^2}$$
$$y'' = \frac{2}{x^3}$$
$$y''' = -\frac{2 \cdot 3}{x^4}$$
$$y^{(4)} = \frac{2 \cdot 3 \cdot 4}{x^5}$$
$$y^{(n)} = \frac{(-1)^n(2 \cdot 3 \cdot 4 \cdots n)}{x^n} = (-1)^n \cdot \frac{n!}{x^n}$$

13. We have $y' = dy/dx$ and $u' = du/dx$

$$y' = f'(gh) \cdot (g'h + gh')$$
$$y'' = f''(gh) \cdot (g'h + gh')^2 + f'(gh) \cdot (g''h + 2g'h' + gh'')$$

$$\begin{aligned} y''' = {} & f'''(gh) \cdot (g'h + gh')^3 \\ & + 2(g'h + gh')(g''h + 2g'h' + gh'') \cdot f''(gh) \\ & + f''(gh)(g'h + gh')(g''h + 2g'h' + gh'') \\ & + f'(gh)(g'''h + g''h' + 2g''h' + 2g'h'' + g'h'' + gh''') \end{aligned}$$