Newton's Method

Newtons method uses the idea that around the point $x = z_0$ a differentiable function looks like a line (linearization) and so, that tangent line could be used to estimate the zero and that zero would give us a new point $x = z_1$ that we could now look around and repeat. The linearization of a function is just the tangent line!

$$f(x) \approx L_k(x) = f(z_k) + f'(z_k)(x - z_k)$$

Since we find the next guess of zero by finding the zero of L_k we get that

$$0 = f(z_k) + f'(z_k)(x - z_k)$$

$$x = -\frac{f(z_k)}{f'(z_k)} + z_k = z_{k+1}$$

We can use our calculator to repeated evaluate this expression by using the "ANS" key.

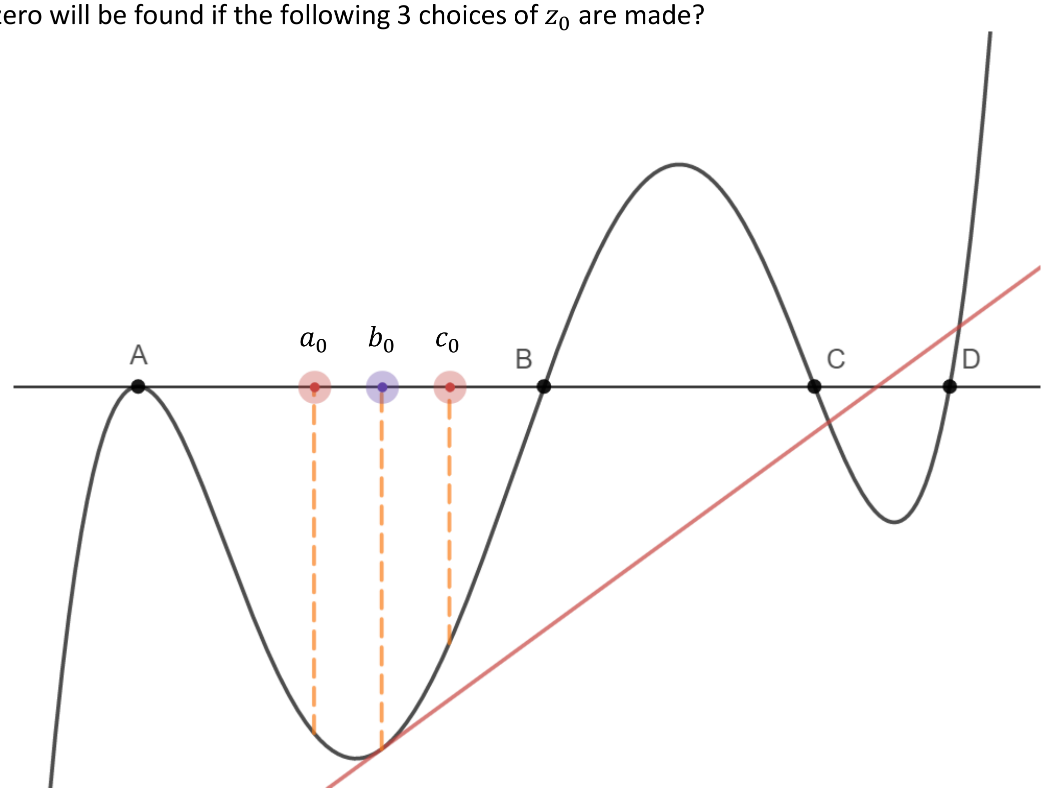
Be aware that Newton's method is very sensitive at $x = z_0$ if $f'(z_0) \approx 0$ and it may hover around a point that is very close to a zero but not actually.

Things we need to know and understand:

- Understands how Newton's method approximates the zeros
- How to use Newton's method to confidently find all the zeros in given or reasonable domain.
- How to use Newton's method quickly to find some zeros

Review Questions:

1. Find the zeros of the curve $y = x^3 - 3x + 1$
2. Find the zeros of the curve $y = x^4 - 3x^2 + x + 1$
3. What zero will be found if the following 3 choices of z_0 are made?



Solutions: Let $A = \text{ANS}$

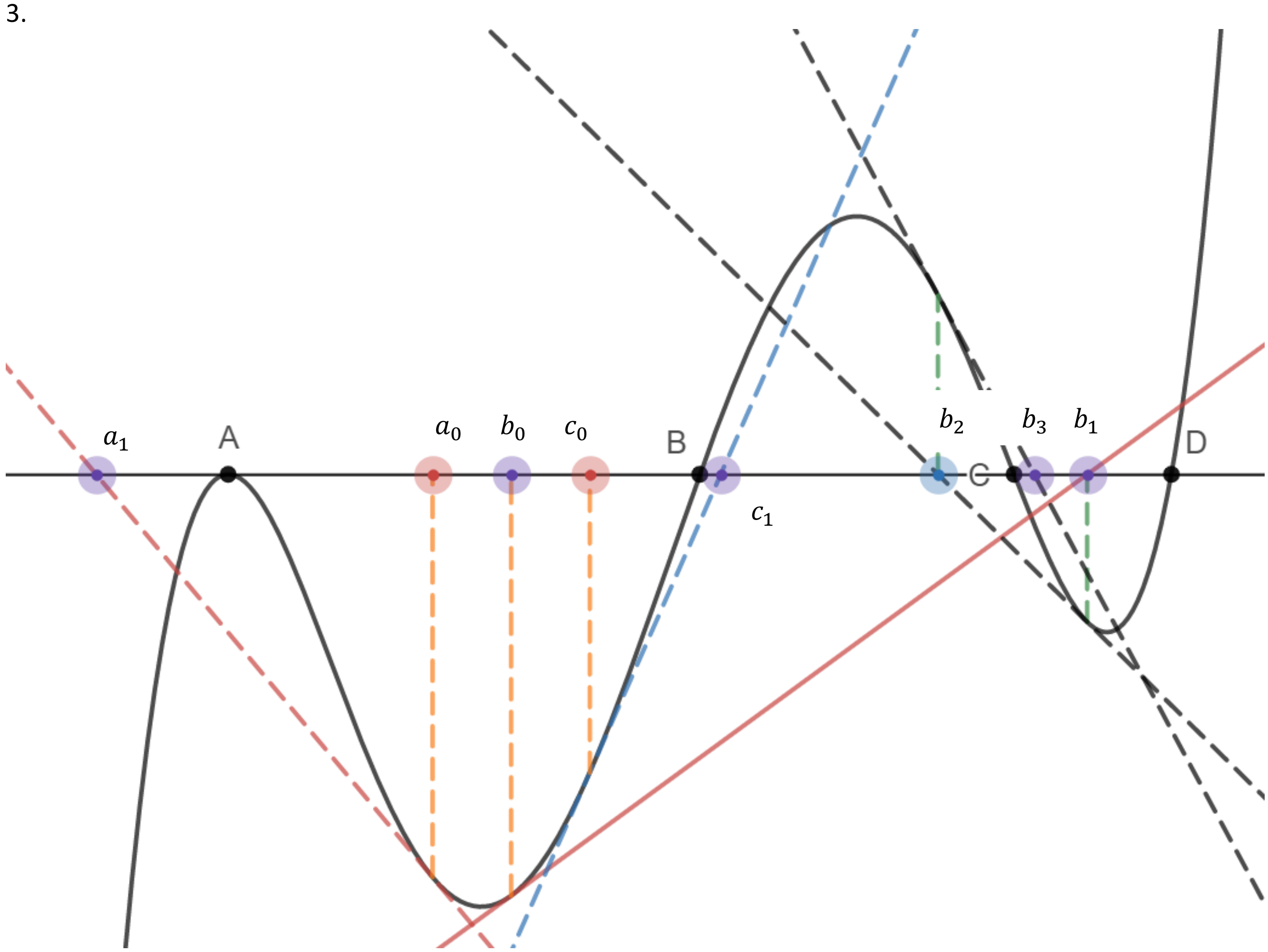
1. Solve for $x = -\frac{A^3-3A+1}{3A^2-3} + A$

   If $z_0 = 0 \rightarrow$ find $x = 0.347 \ldots$
   If $z_0 \geq 2 \rightarrow$ find $x = 1.532 \ldots$
   If $z_0 \leq -2 \rightarrow$ find $x = -1.879 \ldots$
   Note that if $z_0 = \pm 1$ then the slope is 0 and we don't find another zero.

2. Solve for $x = -\frac{A^4-3A^2+A+1}{4A^3-6A+1} + A$

   If $z_0 = -1, 0 \rightarrow$ find $x = -0.445 \ldots$
   If $z_0 = 1 \rightarrow$ find $x = 1$
   If $z_0 \geq 2 \rightarrow$ find $x = 1.247 \ldots$
   If $z_0 \leq -2 \rightarrow$ find $x = -1.802 \ldots$

3.



So a_0 will find A, b_0 will find C, and c_0 will find B

First Derivative Test

To identify when a function has a maximum or a minimum, we saw that these values occur when the function, f, changes direction (goes up then down or vice versa). For a differentiable function this means that the derivative will go from positive to 0 and then to negative. Otherwise there will be a corner where the derivative goes from positive to undefined to negative.

We talked about two types of extrema: absolute and local. Local extrema are the biggest or smallest value in some neighbourhood (a ball around the point) which included endpoints. Absolute extrema are the biggest or smallest value in the domain of the function.

The first derivative test tells you when you have a maximum or a minimum. If a function goes from increasing ($f' > 0$) to decreasing ($f' < 0$) then f' changes sign from positive to negative, so we have a local maximum (up and then down), and vice versa for decreasing to increasing getting a local minimum. NOTE: it is not sufficient to check if $f' = 0$. We may have a point where the function does not change sign and thus does not change direction.

Things we need to know and understand:

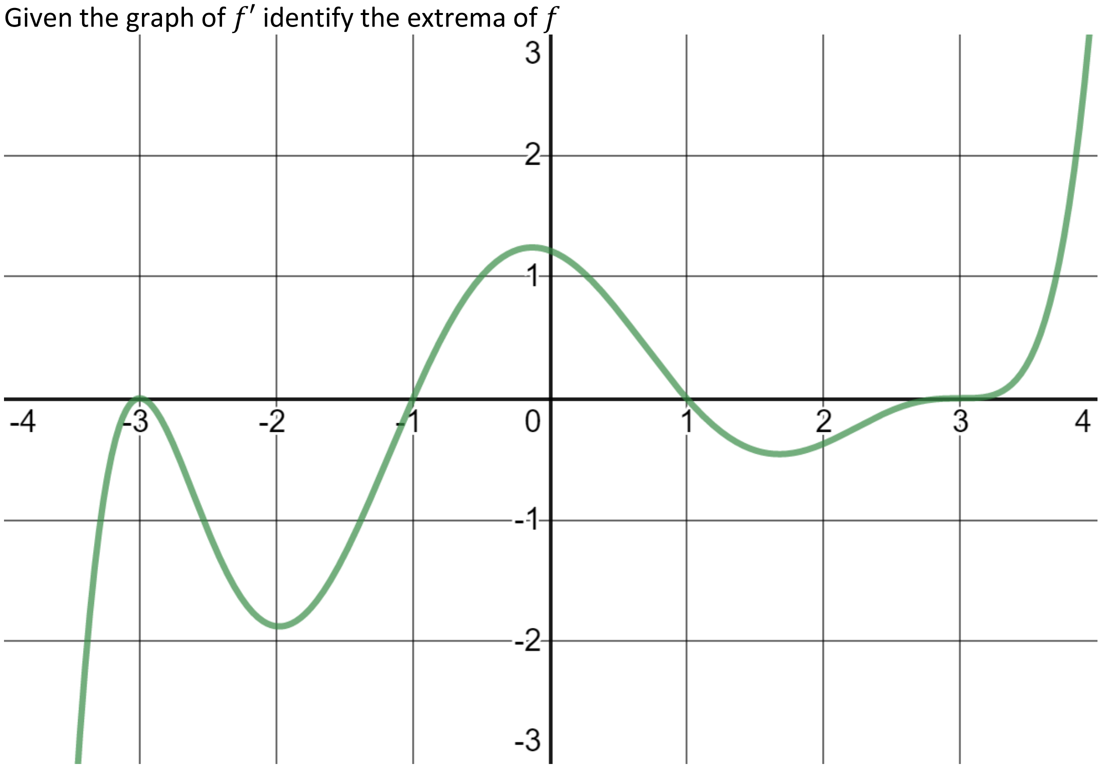
- How to use first derivative test given a table or graph of f' to determine information about the extrema of f
- How to determine if an extremum is a local or absolute and if it is maximum or minimum.

Review Questions:

4. Find the extrema of $f(x) = x^4 - 2x^3 - 2x^2$ on the interval $[-1, 3]$.
5. Find the extrema of $f(x) = x^4 - 4x^3 - 6x - 2$
6. Find the extrema of $f(x) = \frac{x^5}{5} - x^3 + x^2 - 2$
7. Given the values of $f'(x)$ below, identify all maximums and minimums

x	$x < -1$	-1	$-1 < x < 2$	2	$2 < x < 6$	6	$x > 6$
$f'(x)$	Positive	0	Positive	undefined	Negative	0	Positive

8. Given the graph of f' identify the extrema of f



Solutions:

4. We have that $f'(x) = 4x^3 - 6x^2 - 4x = 2x(2x^2 - 3x - 2) = 2x(2x+1)(x-2)$. So we have critical points at $x = -0.5$, 0, and 2

x		-0.5		0		2	
$f'(x)$	Neg	0	Pos	0	Neg	0	Pos
Shape	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$
Type		Min		Max		Min	

$f(-1) = 1$ Local Max
$f(-0.5) = -0.1875$ Local Min
$f(0) = 0$ Local Max
$f(2) = -8$ Absolute Min
$f(3) = 9$ Absolute Max

5. We have that $f'(x) = 4x^3 - 12x^2 - 6$ and we can find the zeros using Newton's method.

$$x = -\frac{4A^3 - 12A^2 - 6}{12A^2 - 24A} + A$$

If $-1.5 \le z_0 \le -0.5 = 0 \rightarrow$ find $x = -0.832$ ...
If $z_0 > 0 \rightarrow$ find $x = 0.642$ ...
If $z_0 < -2 \rightarrow$ find $x = -2.810$ ...

x		-2.8		-0.8		0.6	
$f'(x)$	Neg	0	Pos	0	Neg	0	Pos
Shape	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$
Type		Min		Max		Min	

$f(-2.810) = -11.544$ Absolute Min
$f(-0.832) = 1.167$ Local Max
$f(0.642) = -4.624$ Local Min

6. We have that $f'(x) = x^4 - 3x^2 + 2x = x(x^3 - 3x + 2)$ and we can factor the cubic since $x = 1$ is clearly a zero. So $f'(x) = x(x-1)(x^2 + x - 2) = x(x-1)(x-1)(x+2)$, and there are zeros at $x = -2, 0, 1$

x		-2				0				1	
$f'(x)$	Pos	0		Neg		0		Pos		0	Pos
Shape	$\nearrow$			$\searrow$				$\nearrow$			$\nearrow$
Type		Max				Min				Neither	

$f(-2) = 3.6$ Local Max
$f(0) = -2$ Local Min

7. Local max at $x = 2$ and a local min at $x = 6$
8. Local min at $x = -1$; local max at $x = 1$; and local min at $x = 3$

Optimization

A lot like related rates, our objective is to find the relationship that connects some independent variable to a dependent variable. When dealing with geometric objects, our goal is typically to maximize or minimize the volume, area, or perimeter (distance). Use triangles and Pythagoras to measure distances.

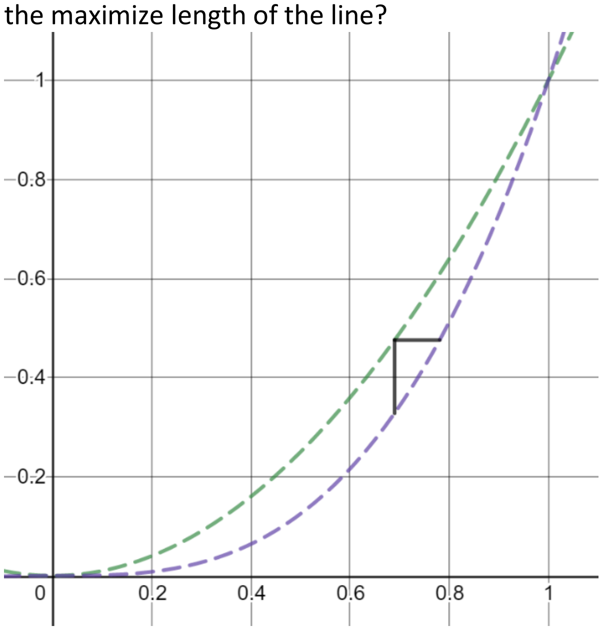
Think to yourself what the parameters are that you have free choice about and what is then determined from that result. That result will depend on your earlier choice and you can express it in terms of the other variable.

Things we need to know and understand:

- How to draw a model for a problem
- How to identify triangles inside circles and find the distance between points.
- Profit = Revenue – Cost
- Revenue = quantity × price
- Average Cost = Cost ÷ quantity

Review Questions:

9. A cone with an open top has a fixed volume but the material to make it should be minimized. What is the optimal dimensions of a cone to minimize material? The surface area of a cone is $A = \pi r\sqrt{r^2 + h^2}$
10. What are the dimensions of the largest rectangle that can be inscribed between the curve $y = \frac{1}{x^2+1}$ and the x-axis?
11. On the interval $[0, 1]$ a bent line is made between the curves $y = x^2$ and $y = x^3$ as shown below. What is the maximize length of the line?



12. It costs a firm $C(q)$ dollars to produce q grams per day of a certain chemical, where
$$C(q) = 1000 + 0.1\left(\frac{q}{100}\right)^3$$
The firm can currently sell 2000 grams at \$6/gram, and they know that if they lower the price \$1/gram they will sell 500 grams more (and vice versa if they raise the price).
    a. Determine the quantity they should produce to minimize the average cost.
    b. Write an expression for demand in terms of price.
    c. Determine the price they should sell at to maximize the revenue.
    d. Write an expression for revenue in terms of quantity and determine the quantity they should produce to maximize the profit.

Solutions:

9. $V=\frac{1}{3}\pi r^2 h$ and so $h=\frac{3V}{\pi r^2}$ Substitute into surface area $A=\pi r\sqrt{r^2+h^2}=\pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}}=\sqrt{\pi^2 r^4+\frac{9V^2}{r^2}}$

$$A'(r)=\frac{\left(4\pi^2 r^3-\frac{18V^2}{r^3}\right)}{2\sqrt{\pi^2 r^4+\frac{9V^2}{r^2}}}=0\Rightarrow\left(4\pi^2 r^3-\frac{18V^2}{r^3}\right)=0$$
$$4\pi^2 r^6=18V^2$$
$$r^6=\frac{1}{2}\left(\frac{3V}{\pi}\right)^2\Rightarrow r=\sqrt[3]{\frac{3V}{\sqrt{2}\pi}}$$
$$\Rightarrow h=\frac{3V}{\pi}\cdot\sqrt[3]{\frac{2\pi^2}{9V^2}}=\sqrt[3]{\frac{6V}{\pi}}$$

10. Dimensions are $2x$ by $y=\frac{1}{x^2+1}$ so $A(x)=\frac{2x}{x^2+1}$
$$A'(x)=\frac{2(x^2+1)-4x^2}{x^2+1}=0$$
$$\Rightarrow 2x^2+2-4x^2=0$$
$$x^2=1$$
$$x=1$$
So the rectangle will be 2 by 1/2. Note that for $A(x)$ the domain is $x\in\mathbb{R}$ so we could have a rectangle with as large a width as we want, but the height will get very small very quick.

11. Let $L(x)$ be the length of the line at position x. Then it goes x^2-x^3 units up and it will be at coordinate (x,x^2) and connect to a point at the same height (a,x^2) on $y=x^3\Rightarrow x^2=a^3\Rightarrow a=\sqrt[3]{x^2}$ so the line goes $\sqrt[3]{x^2}-x$ units right.
$$L(x)=x^2-x^3+x^{\frac{2}{3}}-x$$
$$L'(x)=2x-3x^2+\frac{2}{3}x^{-\frac{1}{3}}-1=0$$
Use Newton's method to find zeros
$$x=-\frac{2A-3A^2+\frac{2}{3}A^{-\frac{1}{3}}-1}{2-6A-\frac{2}{9}A^{-\frac{4}{3}}}+A$$
I expect L is maximized around 0.5 so I use that as z_0, and I find $x=0.553\ldots$ (in fact any choice of z_0 in $(0,1)$ will find this zero as it is the only one). We have $L(0.553)=0.257$ units long.

12.
   a. $\frac{d}{dq}\left(\frac{C(q)}{q}\right)=\frac{d}{dq}\left(\frac{1000}{q}+\frac{q^2}{10^7}\right)=-\frac{10^3}{q^2}+\frac{2q}{10^7}=0$
   $\Rightarrow 2q^3=10^{10}\Rightarrow q=1710$ grams
   b. $q(p)=-500(p-6)+2000$
   c. $R(p)=p\cdot q(p)=-500p^2+5000p$
   $$R'(p)=-1000p+5000=0$$
   $$\Rightarrow p=\$5$$
   Which will make the quantity produced 2500 grams
   d. $R(q)=q\cdot p(q)=q\left(10-\frac{q}{500}\right)$ and we have that $P(q)=R(q)-C(q)$
   $$P'(q)=0\Rightarrow R'(q)=C'(q)$$
   $$\Rightarrow 10-\frac{2q}{500}=3\left(\frac{q}{100}\right)^2\cdot\frac{1}{1000}$$
   $$10^4-4q=\frac{3q^2}{10^4}$$
   $$3q^2+4\cdot 10^4 q-10^8=0\Rightarrow q=2153\text{ grams}$$