Horizontal and Slant Asymptotes

When finding slant and horizontal asymptotes of a function f, we are looking for a simpler function s, (usually linear) such that as x grows large f and s become the same value.

$$\lim_{x\to\infty}\left(f(x)-s(x)\right)=0$$

Note that this means that $f(x)\approx s(x)$ once x gets large enough.

To simplify a rational function, we have basically 3 cases.

1. Numerator degree < denominal degree ⇒ the horizontal asymptote is $y=0$ because the bottom grows faster than the top.
$$f(x)=\frac{\cancel{a_n x^n+a_{n-1}x^{n-1}+\cdots}}{b_m x^m+b_{m-1}x^{m-1}+\cdots},\qquad n<m$$
$$s(x)=0$$
2. Numerator degree $=$ denominator degree ⇒ the horizontal asymptote is $y=c$ because they grow at the same rate.
$$f(x)=\frac{a_n x^n+\cancel{a_{n-1}x^{n-1}+\cdots}}{b_n x^n+\cancel{b_{n-1}x^{n-1}+\cdots}}$$
$$s(x)=\frac{a_n}{b_n}$$
3. Numerator degree > denominator degree ⇒ the horizontal asymptote is a line or curve because now the top grows faster than the bottom.
$$f(x)=\frac{a_{n+1}x^{n+1}+a_n x^n+\cancel{a_{n-1}x^{n-1}+\cdots}}{b_n x^n+b_{n-1}x^{n-1}+\cancel{b_{n-2}x^{n-2}+\cdots}}$$
$$s(x)=\frac{a_n}{b_n}x+c$$
In general, if we need k terms from the numerator, we need k terms from the denominator. The exact equation of s is using long division.

Things we need to know and understand:

- Can determine slant and horizontal asymptotes of a rational function.
- Can use slant asymptotes to approximate the function for large values of x
- Can graph a function using its horizontal asymptotes

Review Questions:

1. Find the slant asymptote of
$$f(x)=\frac{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$$
2. Find the slant asymptote of
$$f(x)=\frac{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$$
3. Find the slant asymptote of
$$f(x)=\frac{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{b_3x^3+b_1x+b_0}$$
4. Find the slant asymptote of
$$f(x)=\frac{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{b_3x^3+b_2x^2+b_1x+b_0}$$

5. Determine the value of k so that $y = 3x - 12$ is the slant asymptote of
$$\frac{3x^5 - 4x^4 + 13x^2 - 2}{x^4 + kx^3 - 5x^2 + 13x - 1}$$
6. What is the value of $f(10^{12})$ given
$$f(x) = \left[\frac{48x^7 - 15x^6 + 42x^4 - 300x + 250}{16x^6 + 27x^5 - 32x^3 + 55x^2 - 4000}\right]$$
Here, $[x]$ rounds x to the nearest integer.

Solutions:
1. $s(x) = 0$
2. $s(x) = \frac{a_4}{b_4}$
3. $s(x) = \frac{a_4}{b_3}x + \frac{a_3}{b_3}$
4. $s(x) = \frac{a_4}{b_3}x + \frac{a_3}{b_3} - \frac{a_4}{b_3} \cdot \frac{b_2}{b_3}$
5. $k = 8/3$
6. $f(10^{12}) = 3 \cdot 10^{12} - 6 = 2\,999\,999\,999\,994$

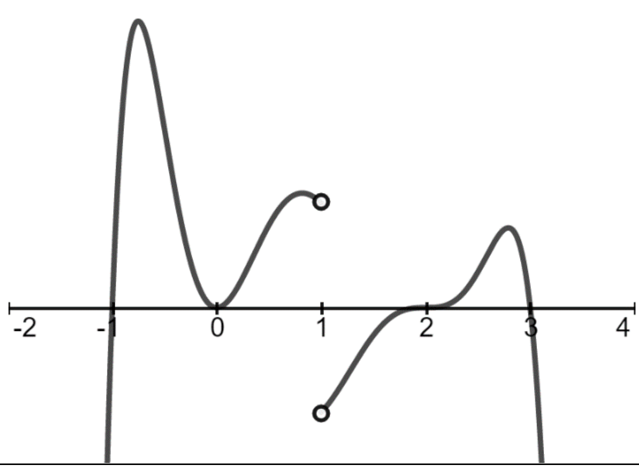
Concavity

We identified the second derivative, f'' or $\frac{d^2f}{dx^2}$, as the rate of change of the slope. So if $f'' > 0$, then the slope is getting more positive (like a " ∪ " shape), and if $f'' < 0$ then the slope is becoming more negative (like a " ∩ " shape). We call this concave up and down respectively. Anytime we switch concavity we have an inflection point.

We also saw that when we are concave up, $f'' > 0$, we have the potential for a local minimum if $f' = 0$ in that region, and conversely if $f'' < 0$ and $f' = 0$ we are looking at a local maximum. This is the second derivative test.

Things we need to know and understand:
- How to use second derivative test given an equation, table or graph of f'' and zeros of f' to determine information about the extrema of f
- How to find inflection points given an equation, table or graph of f''

Review Questions:
7. Find the inflection points and intervals of concavity of $y = \frac{1}{x^2+1}$
8. Find the inflection points and intervals of concavity of $y = \frac{3}{10}x^5 - x^4 + x^3 - x + 2$
9. Find the inflection points given the graph of f'' below (f is continuous). If $f'(x) = 0$ when $x = -1.5,\ 0,\ 3.5$ and $f'(1)$ is defined and positive, where are the local extrema and what are the type?



10. Given the table of value of f'' find the inflection points of f (assuming it is continuous).

x	$x < 0.5$	$x = 0.5$	$0.5 < x < 2$	$x = 2$	$2 < x < 4.5$
$f''(x)$	Positive	0	Negative	0	Positive

$x = 4.5$	$4.5 < x < 7$	$x = 7$	$7 < x < 7.5$	$x = 7.5$	$x > 7.5$
0	Negative	Undefined	Positive	0	Negative

If we also know that $f'(x) = 0$ when $x = 0, 2, 6,$ and 8 and $f'(7)$ is undefined, where are the extrema of f and what is the type? IMPORTANT: There are no other times f' is 0 or undefined.

Solutions:

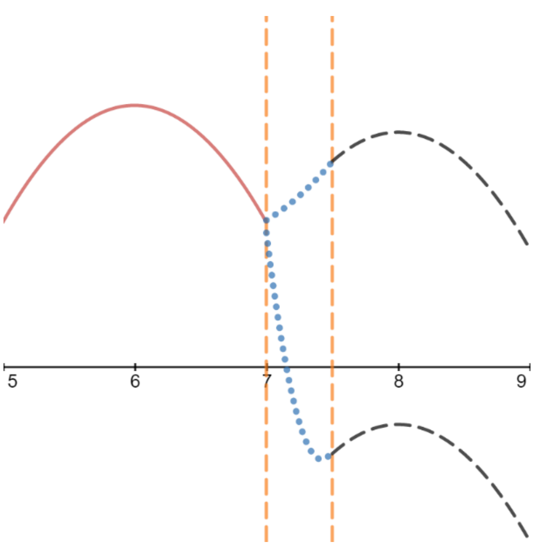
7. Using power rule, $y = (x^2+1)^{-1}$ so $y' = -\frac{2x}{(x^2+1)^2}$ and $y'' = 0$ when $2(x^2+1)^2 - 8x^2(x^2+1) = 0$ (the numerator of y''). Factor and solve for x.

$$2(x^2+1)(x^2+1-4x^2) = 0$$
$$1 - 3x^2 = 0$$
$$x = \pm\frac{1}{\sqrt{3}}$$

We have y'' is a parabola with two zeros and switches sign over the zeros so y is concave up when $|x| > \frac{1}{\sqrt{3}}$ and concave down when $|x| < \frac{1}{\sqrt{3}}$

8. We have $y' = \frac{3}{2}x^4 - 4x^3 + 3x^2 - 1$ and $y'' = 6x^3 - 12x^2 + 6x = 6x(x^2 - 2x + 1) = 6x(x-1)^2 = 0$ which means $x = 0\,, 1$ BUT y'' does not change sign at $x = 1$ so the only inflection point is when $x = 0$. As y'' is a cubic it starts negative so y is concave down if $x < 0$ and concave up when $x > 0, x \neq 1$. Note that y is not concave up or down at $x = 1$ since it has concavity of 0.
9. Change concavity at $x = -1,\ 1,\ 2,$ and 3. There should be a local maximum at $x = -1.5$ and $x = 3.5$. When using the second derivative test at $x = 0$ we have it be incoclusive since $f''(0) = 0$. However, the concavity stays concave up around $x = 0$ which means we should have a local minimum.
10. Change sign at $x = 0.5,\ 2,\ 4.5,\ 7,\ 7.5$. For extrema organize your points and concavity

x	0	2	6	7	8
Shape	∪	∩ to ∪	∩	∩ to ∪	∩
Extrema	Min	None, because slope is 0 and we change concavity	Max	Min** See below	Max



Standard for $x = 0, 2, 6, 8$, but at $x = 7$ we need to connect two concave down regions together with a concave up region AND have it be smooth (differentiable) at $x = 7.5$.

Shown on the left are two ways to do this. One way has no slope of zero between them while the other does. There is no point in the interval that does have a slope of 0 so it must look like the top graph. Hence a minimum at $x = 7$

Curve Sketching

When curve sketching, we just bring everything together and find the zeros by factoring or using Newton's method, find the extrema by analyzing f' and the concavity by analyzing f''. If the function is a rational function we should also investigate asymptote behaviour as $x \to \pm\infty$.

Things we need to know and understand:

- How to differentiate f, f', f'' so we can find the relevant information.
- When to use Newton's method

Review Questions:

11. Sketch the curve in #7
12. Sketch the curve in #8

Solutions: (graphs at the bottom)

11. Since we know $y' = 0$ when $x = 0$ and is concave down there is a maximum at $x = 0$. The function has no zeros but it does have a horizontal asymptote of $y = 0$

Recap:

Zeros	Extrema	Inflection Points
None, Horiz. Asymptote $y = 0$	$x = 0$	$x = \pm\frac{1}{\sqrt{3}}$

12. We have $y' = \frac{3}{2}x^4 - 4x^3 + 3x^2 - 1$ and to find it's zeros I use Newton's method.

$$x = -\frac{\frac{3}{2}A^4 - 4A^3 + 3A^2 - 1}{6A^3 - 12A^2 + 6A} + A$$

We should find $x = 1.560\ldots; -0.444\ldots$

x	$x < -0.444$	-0.444	$-0.4 < x < 1.6$	1.560	$x > 1.560$
y'	Positive	0	Negative	0	Positive

We find zeros of y using Newton's method

$$x = -\frac{0.3A^5 - A^4 + A^3 - A + 2}{1.5A^4 - 4A^3 + 3A^2 - 1} + A$$

We should find $x = -1.082\ldots$

Recap:

Zeros	Extrema	Inflection Points
$x = -1.082\ldots$	$x = -0.444\ldots\ (MAX)$ $x = 1.560\ldots\ (MIN)$	$x = 0$ At $x = 1$ the curve gets flat