Derivative of e^x and $\ln x$

We saw using the limit properties that

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h}$$

And

$$L = \lim_{h\to 0}\frac{e^h-1}{h} \stackrel{\text{def}}{=} 1$$

The value of e that makes the limit equal to 1 is 2.718281828… and we call this Euler's Number. By definition we have that

$$\frac{d}{dx}e^x = e^x$$

And so if we want to differentiate $y = \ln x$ we can change to exponential

$$e^y = e^{\ln x} = x$$
$$\Rightarrow e^y \cdot \frac{dy}{dx} = 1$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

We must remember chain rule, so if $f(x) = e^x$ and we want to find dy/dx from

$$y = e^u$$

Then we have to find df/dx where

$$f(u) = e^u = y$$

We know that

$$\frac{dy}{dx} = f'(u)\cdot\frac{du}{dx} = e^u\cdot\frac{du}{dx}$$

So, chain rule continues through the power.

We can also do this any base since

$$b^x = e^{\ln b^x} = e^{x\cdot\ln b}$$
$$\Rightarrow \frac{d}{dx}b^x = \frac{d}{dx}e^{x\cdot\ln b} = e^{x\cdot\ln b}\cdot\ln b = b^x\ln b$$

Likewise, we find

$$\frac{d}{dx}\log_b x = \frac{1}{x\cdot\ln b}$$

With these new derivatives we should be able to use derivative rules and take the derivative of any product, quotient or chain we are given.

Note! The function $y = \ln(\ln x)$ is a composition of functions. We are not multiplying by ln because that does not make sense. What is $\ln x$. It is a function that maps $x \to y$ such that $e^y = x$. So $\ln(\ln x) = y$ maps

$$x \to z \to y$$

Such that $x = e^z$ and $z = e^y$

Just as $\sqrt{\sqrt{x+2}}$ does not multiply square roots together. Because writing $\sqrt{}\cdot\sqrt{}$ make no sense at all.

Finally, we can simplify logs using log laws

$$\ln(ab) = \ln a + \ln b$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\ln a^n = n \cdot \ln a$$

When we have a big product or quotient, we can take the log of both sides and find the derivative implicitly.

Things we need to know and understand:

- How to take derivative of exponential and log function with any base
- Can use product, quotient and chain rule with exponential and log functions.
- Can simplify logs using log laws
- Can use simple applications with exponential and logs such as Newton's method and finding extrema's and concavity

Review Questions:

For the following find dy/dx

1. $y = b^u$
2. $y = \log_b u$
3. $y = \dfrac{x^2 + 3e^x}{2e^x - x}$
4. $y = 3z^2 e^w$
5. $y = \left(5^{4\sqrt{x}+x^2}\right)^3$
6. $y = \ln(\ln(\ln(x^2 + z))$
7. $y = \ln\sqrt{\dfrac{(x^2+1)^5}{(3x+2)^{20}}}$
8. $y = \sqrt{\log\sqrt{t}}$
9. $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$
10. $y = \left(\dfrac{x(x-2)\ln(x+3)}{(x^2-1)e^{2x}}\right)^3$
11. $y = \sqrt{x}^{x^2}$
12. $y = \ln^{\ln x} x$
13. $y = x^{\ln^x x}$

Find the solution to the following

14. $e^x - \ln x = 4$ (one solution)
15. $\ln(\ln x) = 5 - x$ (one solution)

Graph the following functions

16. $y = x \cdot \ln(x^2)$ Note that $\lim_{x \to 0} x \ln(x^2) = 0$
17. $y = x^2 e^{-x}$

Solutions:

1. $b^u \cdot \ln b \cdot \frac{du}{dx}$
2. $\frac{1}{u \cdot \ln b} \cdot \frac{du}{dx}$
3. Quotient rule ⇒
$$\frac{(2x + 3e^x)(2e^x - x) - (2e^x - 1)(x^2 + 3e^x)}{(2e^x - x)^2}$$
4. Product and chain rule ⇒
$$6ze^w \cdot \frac{dz}{dx} + 3z^2 e^w \cdot \frac{dw}{dx}$$
5. Chain rule ⇒
$$3\left(5^{4\sqrt{x}+x^2}\right)^2 \cdot \left(5^{4\sqrt{x}+x^2} \cdot \ln 5\right) \cdot \left(\frac{2}{\sqrt{x}} + 2x\right)$$
6. Chain rule ⇒
$$\frac{1}{\ln(\ln(x^2 + z))} \cdot \frac{1}{\ln(x^2 + z)} \cdot \left(\frac{1}{x^2 + z}\right) \cdot \left(2x + \frac{dz}{dx}\right)$$
7. Split into a difference ⇒
$$y = \frac{5}{2}\ln(x^2 + 1) - 10\ln(3x + 2)$$
$$y' = \frac{5x}{x^2 + 1} - \frac{30}{3x + 2}$$
8. Chain rule ⇒
$$\frac{1}{2\sqrt{\log \sqrt{t}}} \cdot \frac{1}{2t \cdot \ln 10} \cdot \frac{dt}{dx}$$
9. Take ln of both sides ⇒
$$\ln y = \frac{1}{3}(\ln x + \ln(x + 1) + \ln(x - 2) - \ln(x^2 + 1) - \ln(2x + 3))$$
$$\frac{1}{y}y' = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x + 1} + \frac{1}{x - 2} - \frac{2x}{x^2 + 1} - \frac{2}{2x + 3}\right)$$
$$y' = \frac{1}{3} \cdot \sqrt[3]{\frac{x(x + 1)(x - 2)}{(x^2 + 1)(2x + 3)}} \cdot \left(\frac{1}{x} + \frac{1}{x + 1} + \frac{1}{x - 2} - \frac{2x}{x^2 + 1} - \frac{2}{2x + 3}\right)$$
10. Take ln of both sides ⇒
$$\ln y = 3(\ln x + \ln(x - 2) + \ln(\ln(x + 3)) - \ln(x^2 - 1) - \ln e^{2x})$$
$$\frac{1}{y}y' = 3\left(\frac{1}{x} + \frac{1}{x - 2} + \frac{1}{\ln(x + 3)} \cdot \frac{1}{x + 3} - \frac{2x}{x^2 - 1} - 2\right)$$
$$y' = 3\left(\frac{x(x - 2)\ln(x + 3)}{(x^2 - 1)e^{2x}}\right)^3 \left(\frac{1}{x} + \frac{1}{x - 2} + \frac{1}{\ln(x + 3)} \cdot \frac{1}{x + 3} - \frac{2x}{x^2 - 1} - 2\right)$$

11. Take ln of both sides ⇒

$$\ln y = x^2 \ln \sqrt{x} = \frac{x^2}{2} \ln x$$
$$\frac{1}{y} y' = x \cdot \ln x + \frac{x}{2}$$

$$y' = x\sqrt{x}^{x^2} \left(\ln x + \frac{1}{2} \right)$$

12. Take ln of both sides ⇒

$$\ln y = \ln x \cdot \ln(\ln x)$$
$$\frac{1}{y} y' = \frac{\ln(\ln x)}{x} + \frac{\ln x}{x \ln x}$$

$$y' = \frac{\ln^{\ln x} x}{x} (\ln(\ln x) + 1)$$

13. Take ln of both sides ⇒

$$\ln y = \ln^x x \cdot \ln x$$

Take ln again ⇒

$$\ln(\ln y) = x \cdot \ln(\ln x) + \ln(\ln x)$$
$$\frac{1}{\ln y} \cdot \frac{1}{y} \cdot y' = \ln(\ln x) + \frac{1}{\ln x} + \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$y' = x^{\ln^x x} \cdot (\ln^x x \cdot \ln x) \cdot \left(\ln(\ln x) + \frac{1}{\ln x} + \frac{1}{\ln x} \cdot \frac{1}{x} \right)$$

14. Set equal to 0 and use Newton's method

$$f(x) = e^x - \ln x - 4 = 0$$
$$f'(x) = e^x - \frac{1}{x}$$

$$x = a - \frac{f(a)}{f'(a)}$$

Find $x = 1.31531485571 \ldots$

15. Set equal to 0 and use Newton's method

$$f(x) = \ln(\ln x) + x - 5 = 0$$
$$f'(x) = \frac{1}{x \ln x} + 1$$

$$x = a - \frac{f(a)}{f'(a)}$$

Find $x = 4.58015087918 \ldots$

16. We need to analyze y' and y''

$$y' = \ln x^2 + 2$$

We find extrema when $\ln x^2 + 2 = 0$

$$\ln x^2 = -2$$
$$\Rightarrow x^2 = e^{-2}$$
$$\Rightarrow x = \pm e^{-1}$$

$$y'' = \frac{2}{x}$$

And y'' is never 0 but $y'' > 0$ when $x > 0$ and vice versa so concave up when $x > 0$ and concave down when $x < 0$. Using second derivative test we know that $x = -e^{-1}$ is a max and $x = e^{-1}$ is a min. We also see there is a zero at $x = \pm 1$

17. Same as above

$$y' = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$$

Which is zero at $x = 0$ and 2

$$y'' = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$$
$$y'' = e^{-x}(2 - 4x + x^2)$$

Which is zero when $x = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

Thus, y is concave up when $x < 2 - \sqrt{2}$ and $x > 2 + \sqrt{2}$ and concave down when $2 - \sqrt{2} < x < 2 + \sqrt{2}$.

Thus $x = 0$ is a min and $x = 2$ is a max