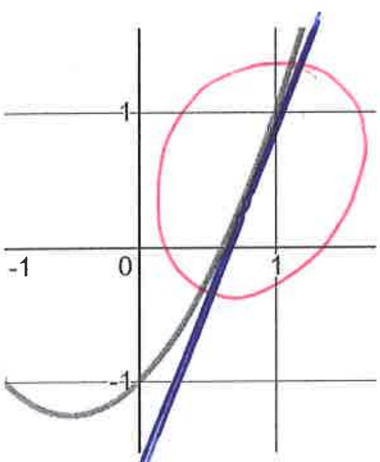
Linearization and Newton's Method

Goal: • Understands linearization is just the tangent line at a point. • Understands that linearization "formula" is just point-slope form of tangent line. • Can use repeated linearization to approximate zeros using your calculator and ANS
Terminology: • Linearization • Newton's Method

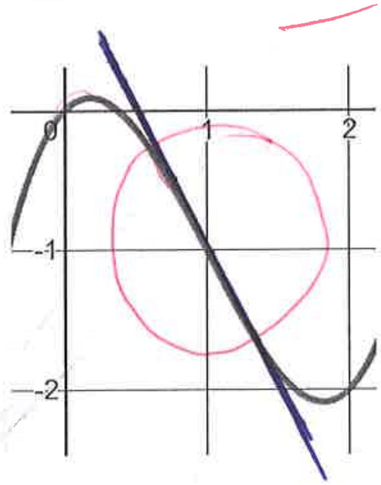
On the board find the equation of the tangent line to the three curves below at $x = 1$ and sketch the curve and its tangent line.

1. $f(x) = x^2 + x - 1, @\, x = 1$



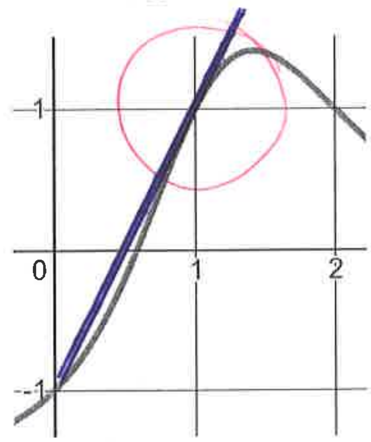
$y = 3(x-1)+1$

2. $g(x) = x^3 - 3x^2 + x, @\, x = 1$



$y = -2(x-1)-1$

3. $h(x) = \dfrac{2x}{x^2 - 2x + 2} - 1,$ $@\, x = 1$



$y = 2(x-1)+1$

What is relevant or what stands out when you compare the tangent line to the original curve?

→ around $x=1$ the tangent line is basically the function curve

What we have done is create a **linearization** of the functions at the point $x = 1$

is just making the equation to the tangent line.

In general if we want to linearize a function at the point $x = a$ we will use point/slope form and have

$y = m(x - x_0) + y_0$ point $(a, f(a))$

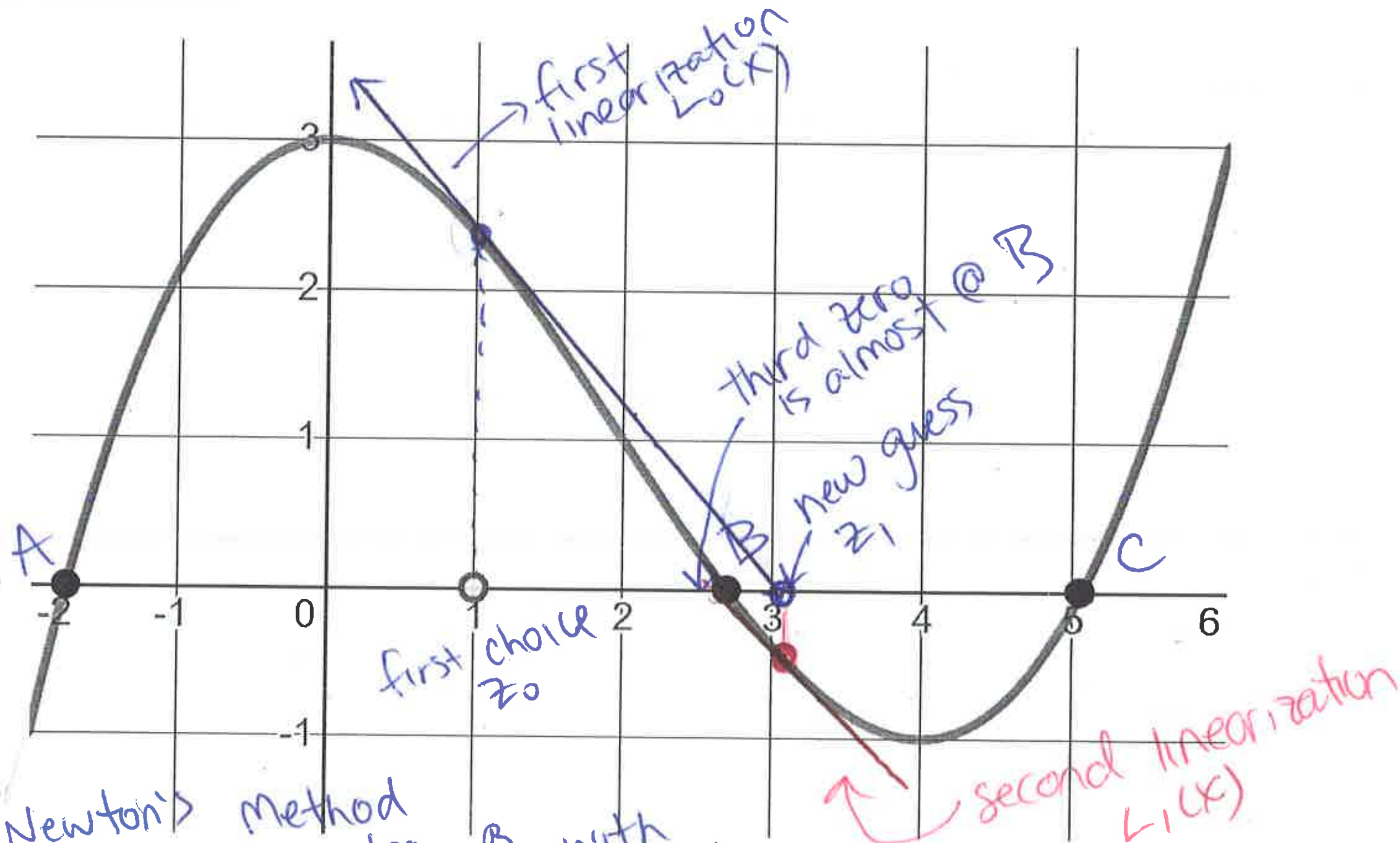
$\Rightarrow L(x) = \underbrace{f'(a)}_{\text{slope}}(x-a) + f(a)$

★ Recursion: doing the same steps repeatedly (loop)

Newton's Method looks to find the zeros of a function by repeatedly using linearization as follows

- Guess the zero to f, call it $z_0 = x$
- Linearize around $x = z_0$, find $L_0(x) = f'(z_0)(x - z_0) + f(z_0)$ → equation of tangent @ z_0
- Find the zero to the linearization, call it z_1
- Repeat and linearize around $x = z_1$, find $L_1(x)$
- Continue until z_n approaches a limit point

What does this look like?



Newton's method approximates B with $z_0 = 1$

So after 3 iterations, we are pretty close to the actual zero.

What if we started with $z_0 = -1$? What is we started with $z_0 = 4$?

Starting $z_0 = -1$ we find A

Starting $z_0 = 4$ is not going to find anything

In general we are solving for the zero of $L_k(x) = f'(z_k)(x - z_k) + f(z_k)$ and then using that to make a new linearization around $x = z_{k+1}$

$$L_k(x) = 0 = f'(z_k)(x - z_k) + f(z_k)$$

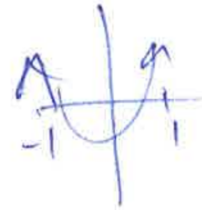
$$x = \frac{-f(z_k)}{f'(z_k)} + z_k = z_{k+1} \leftarrow \text{new guess}$$

$$z_{k+1} = \frac{-f(z_k)}{f'(z_k)} + z_k$$

Example: Find the zeros of $f(x) = x^2 + x - 1$

$f(x) = x^2 + x - 1$ $\qquad$ $f'(x) = 2x + 1$

$ANS^2 + ANS - 1$ $\qquad$ $2ANS + 1$



$$z_{k+1} = \frac{-(x^2 + x - 1)}{(2x+1)} + x$$

Start with $z_0 = 1$, type $\frac{-(ANS^2 + ANS - 1)}{(2ANS + 1)} + ANS$

$\rightarrow$ Find $x = 0.618033989\ldots$

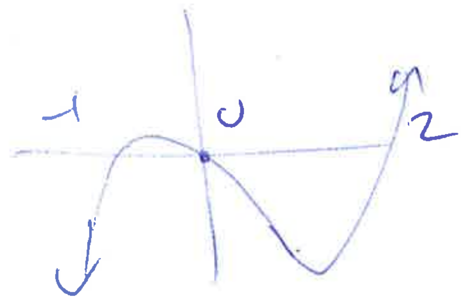
Start with $z_0 = -1$

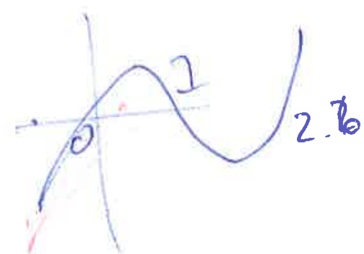
$\rightarrow$ find $x = -1.618033989\ldots$

Practice: Find the zeros of $g(x) = x^3 - 3x^2 + x$

$g'(x) = 3x^2 - 6x + 1$

$$z_{k+1} = \frac{-(x^3 - 3x^2 + x)}{3x^2 - 6x + 1} + x$$





$z_0 = -1$ $\rightarrow$ finds $x = 0$

$z_0 = 2$ $\rightarrow$ finds $x = 2.618033989\ldots$

$z_0 = 1$ $\rightarrow$ finds $x = 0.381966011\ldots$

Practice Problems: 4.5: # 1-3, 8, (4 and 5 are good practice to if you need more)

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- Can use repeated linearization to approximate zeros using your calculator and ANS

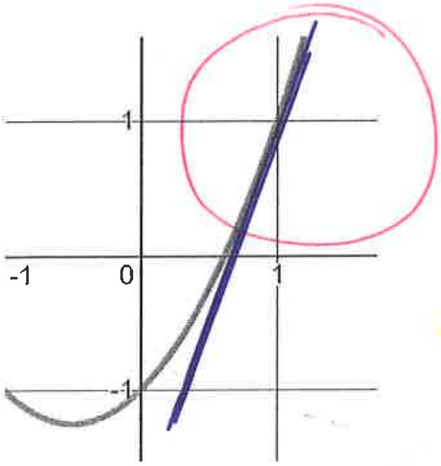
Terminology:

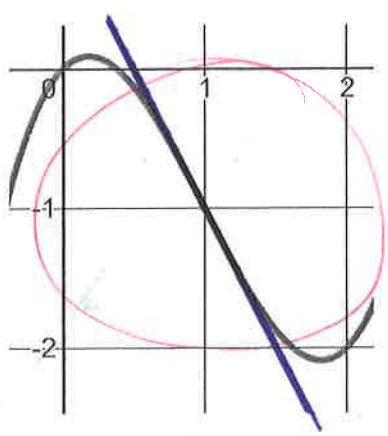
- Linearization
- Newton's Method

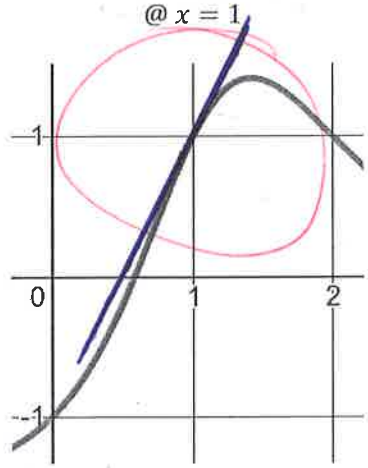
On the board find the equation of the tangent line to the three curves below at $x = 1$ and sketch the curve and its tangent line.

los(h(1.1?)?)?

1. $f(x) = x^2 + x - 1, @\, x = 1$
2. $g(x) = x^3 - 3x^2 + x, @\, x = 1$
3. $h(x) = \dfrac{2x}{x^2 - 2x + 2} - 1,$ $@\, x = 1$







$y = 3(x-1) + 1$ $\sim f(x)$

$y = -2(x-1) - 1$ $\sim g(x)$ near $x = 1$

$y = 2(x-1) + 1$ $\sim h(x)$

$\sin x = x$ $\cos x = 1$

What is relevant or what stands out when you compare the tangent line to the original curve?

The line looks a lot like the curve near $x=1$

What we have done is create a **linearization** of the functions at the point $x = 1$

just the equation of the tangent line.

In general if we want to linearize a function at the point $x = a$ we will use point/slope form and have

$$L(x) = \underbrace{f'(a)}_{m}(x - a) + f(a)$$

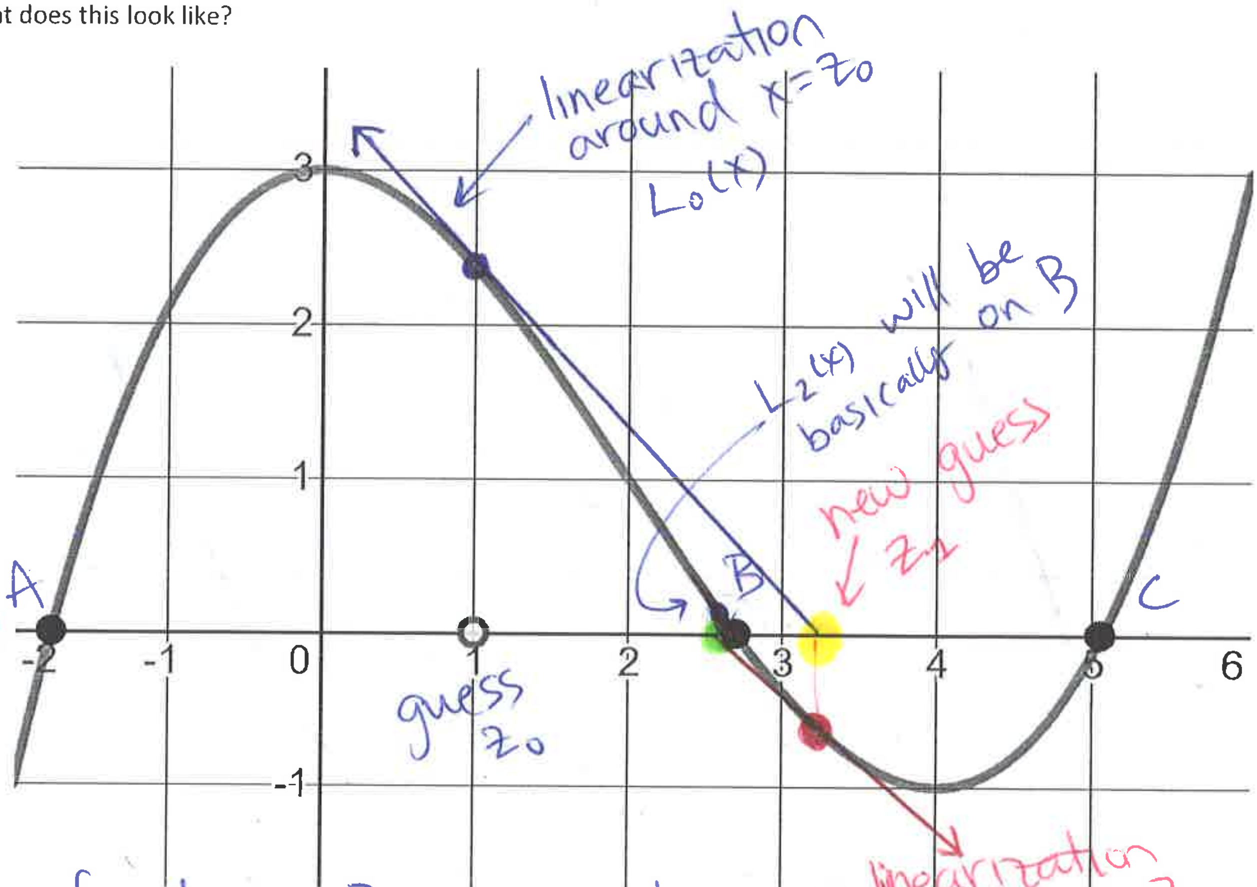
point $(a, f(a))$

☆ Recursion (sp?): Repeatedly doing the same thing (for loop)

Newton's Method looks to find the zeros of a function by repeatedly using linearization as follows

- Guess the zero to f, call it z_0 $= x$
- Linearize around $x = z_0$, find $L_0(x) = f'(z_0)(x - z_0) + f(z_0)$
- Find the zero to the linearization, call it z_1 ← easy
- Repeat and linearize around $x = z_1$, find $L_1(x)$
- Continue until z_n approaches a limit point

What does this look like?



we found $x = B$ as a zero!

So after 3 iterations, we are pretty close to the actual zero.

What if we started with $z_0 = -1$? What is we started with $z_0 = 4$?

with $z_0 = -1$ we find A

with $z_0 = 4$ we find ??? b/c the slope is probably 0

In general we are solving for the zero of $L_k(x) = f'(z_k)(x - z_k) + f(z_k)$ and then using that to make a new linearization around $x = z_{k+1}$

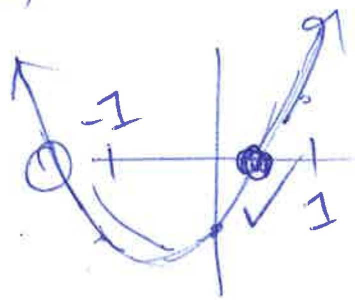
$$L_k(x) = 0 = f'(z_k)(x - z_k) + f(z_k)$$

$$x = -\frac{f(z_k)}{f'(z_k)} + z_k = z_{k+1}$$

↑ next guess

$$z_{k+1} = \frac{-f(z_k)}{f'(z_k)} + z_k$$

Example: Find the zeros of $f(x) = x^2 + x - 1$



$f'(x) = 2x + 1$

$$z_{k+1} = \frac{-(z_k^2 + z_k - 1)}{(2z_k + 1)} + z_k$$

$$x = \frac{-(ANS^2 + ANS - 1)}{(2ANS + 1)} + ANS$$

$z_0 = 1 \rightarrow$ we find $x = 0.618033989\ldots$

$z_0 = -1 \rightarrow$ we find $x = -1.618033989\ldots$

Practice: Find the zeros of $g(x) = x^3 - 3x^2 + x$

$g'(x) = 3x^2 - 6x + 1$

$$z_{k+1} = \frac{-(z_k^3 - 3z_k^2 + z_k)}{3z_k^2 - 6z_k + 1} + z_k$$

$$x = -\frac{(ANS^3 - 3ANS^2 + ANS)}{3ANS^2 - 6ANS + 1} + ANS$$

$z_0 = 1 \rightarrow$ finds $x = 0.381966011\ldots$

$z_0 = 0 \rightarrow$ finds $x = 0$

$z_0 = 3 \rightarrow$ finds $x = 2.618033989\ldots$

Practice Problems: 4.5: # 1-3, 8, (4 and 5 are good practice to if you need more)

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