3. Einstein showed that the mass of an object, m, changes as its velocity changes by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

const

const

Where m_0 is the mass at rest, v is the veloctity, and c is the speed of light (which is constant).

(a) (3 points) Detemine the change in mass with respect to the change in velocity. Show that it always positive. Note $0 < v < c$.

$$\frac{dm}{dv} = ?$$

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$\Rightarrow$$

$$\frac{dm}{dv} = \underbrace{+\frac{m_0}{2}}_{<0}\underbrace{\left(1-\frac{v^2}{c^2}\right)^{-3/2}}_{>0}\underbrace{\left(+\frac{2v}{c^2}\right)}_{<0}$$

$v \ll c$

$$\Rightarrow 0 < \frac{v}{c} < 1$$

>0

(b) (2 points) Momentum of an object with mass m is

$$p = mv$$

Use the definition of m from Einstein above to express the change in p in terms of v over time

$$\frac{dp}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v$$

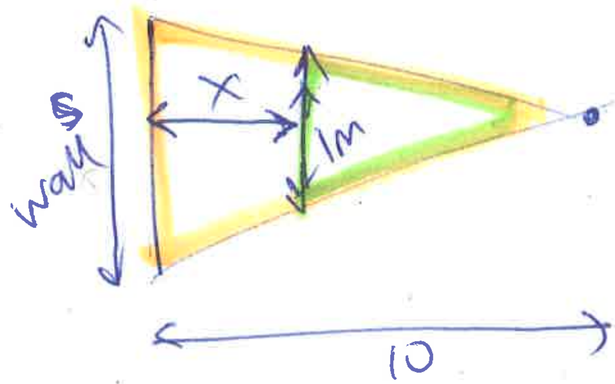
$$\frac{dp}{dt} = v\ldots$$

$$= \boxed{\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dv}{dt} + \frac{m_0 v}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\frac{dv}{dt}}$$

$$\frac{dm}{dt} = \frac{dm}{dv}\cdot\frac{dv}{dt}$$

4. You are standing 10 m in front of a wall with a camera that has a bright light on it. Your friend stands in between you and the wall with their arms stretched out 1 m across. They cast a shadow on the wall that also has their arms stretched out.

(a) (2 points) Draw and label a picture of the scenario above.



$$\frac{1}{10-x} = \frac{s}{10}$$

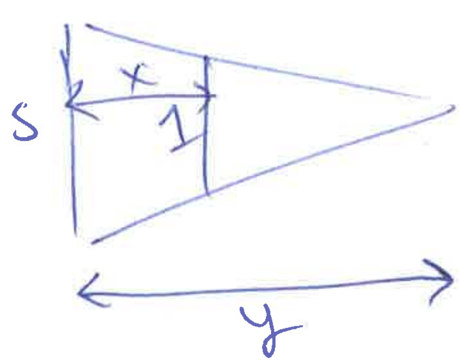
(b) (2 points) If your friend walks toward you at 0.5 m/s, how fast is the length of their shadow growing when they are 2 m in front of the wall?

$$10 = s(10-x) \xrightarrow{\text{let } x=2} s = \frac{10}{8} = 1.25$$

$$\Rightarrow 0 = \frac{ds}{dt}(10-x) + s\left(-\frac{dx}{dt}\right)$$

$$\frac{ds}{dt} = \frac{s\frac{dx}{dt}}{10-x} \Rightarrow \left.\frac{ds}{dt}\right|_{x=2} = \frac{\boxed{1.25}(0.5)}{8} = 0.078 \text{ m/s}$$

(c) (3 points) Adjust your picture and related equation to include you moving toward them at some speed. You do not need to solve the problem again, just make a new related rate equation.

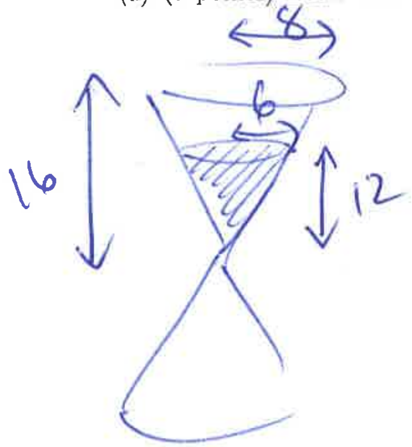


$$\frac{d}{dt}\left(y = s(y-x)\right)$$

$$\frac{dy}{dt} = \frac{ds}{dt}(y-x) + s\left(\frac{dy}{dt} - \frac{dx}{dt}\right)$$

5. Sand is in an hourglass that is two identical cones attached together at their point. Each cone has height of 16 cm and radius of 8 cm. When the hourglass is turned over the sand has a height of 12 cm.

(a) (2 points) Draw and label a picture of the scenario above.



$$\frac{r}{h} = \frac{8}{16} = \frac{1}{2}$$

(b) (1 point) If this truly is an hourglass (all the sand has to leave the top cone in one hour), what is the rate of change of the volume in units of cm^3/min? (Leave π in your answer)

$$V_T = \frac{1}{3}\pi(6)^2(12) = 144\pi \text{ cm}^3$$

$$\frac{dV}{dt} = \frac{144\pi}{60} \frac{\text{cm}^3}{\text{min}} = 2.4\pi \text{ cm}^3/\text{min}$$

(c) (2 points) How fast is the height of the sand in the top cone decreasing when it has a height of 4 cm?

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

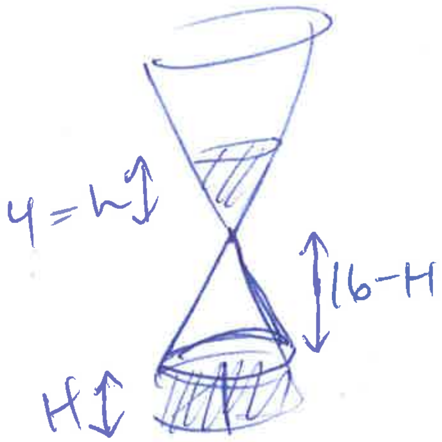
$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = -2.4\cancel{\pi} = \frac{1}{4}\cancel{\pi} h^2 \frac{dh}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=4} = \frac{-2.4(4)}{4^2} = -0.6 \text{ cm/min}$$

(d) (1 point (bonus)) Relate how fast the height of sand in the bottom cone changes to how rate of change of the height of sand in the top cone.

$$\frac{dH}{dt} < \frac{dh}{dt}$$



$$V = \frac{1}{3}\pi\left(\frac{16-H}{2}\right)^2(16-H)$$

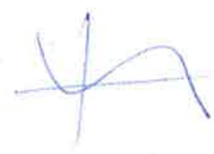
$$V = \frac{1}{12}\pi(16-H)^3$$

$$\frac{dV}{dt} = -\frac{1}{4}\cancel{\pi}(16-H)^2\frac{dH}{dt} \quad 2.4\cancel{\pi}$$

$$\frac{dH}{dt} = \frac{-2.4\times 4}{(16-H)^2}$$

6. (1 point (bonus)) A particle's position function, $x(t)$, satsifies the following

$$\frac{dx}{dt} = x^2(5-x)$$



Describe how the particle moves.

$$\frac{dx}{dt} = 0 \Rightarrow x = 0, 5$$