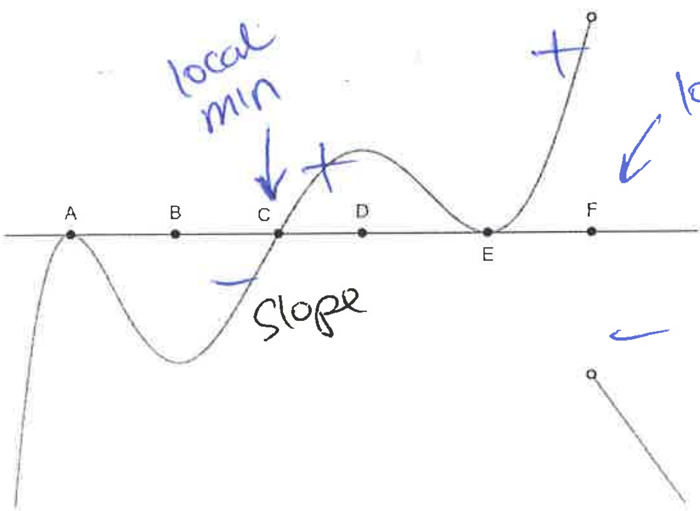
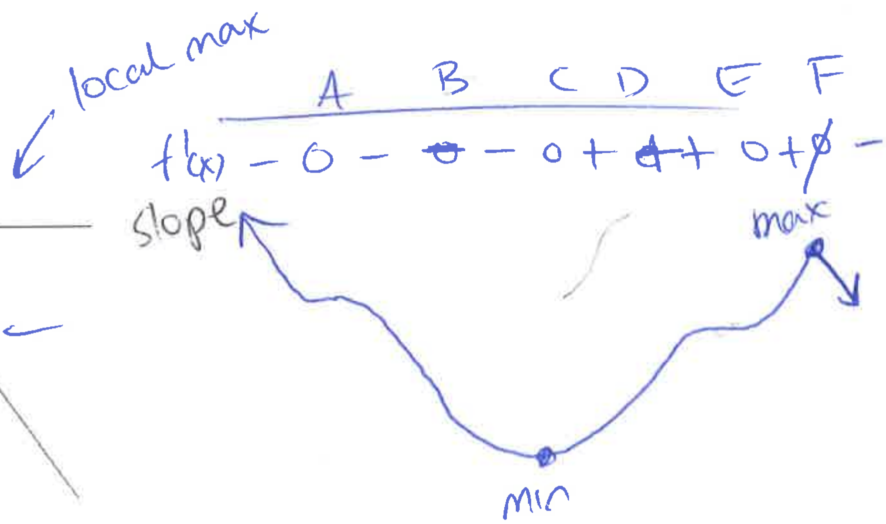
Second Derivative Test and Concavity

Goal: • Can determine the intervals of concavity of a function f by analyzing the function algebraically, or by analyzing the values of f'' given a graph or table. • Can determine the type of an extremum at a point using the second derivative test and concavity.
Terminology: • Concavity (concave up and down) • Inflection Point • Second Derivative Test
Reminder: • Quiz on Tuesday on Concavity and Slant/Horizontal Asymptotes

Review: Given the following graph of $\frac{df}{dx}$, determine the extrema of f and if they are a max or min.



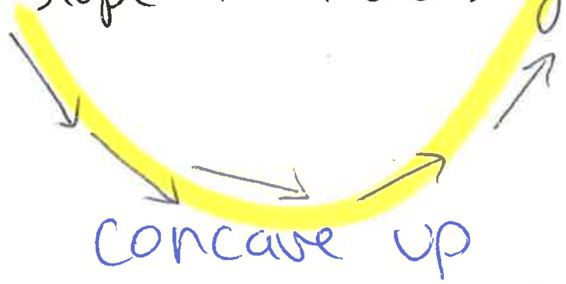


Now that we know what the first derivative tells us (namely the slope and if the curve is going up or down). What does the second derivative tell us?

slope of the slope ⇒ how much the slope is changing

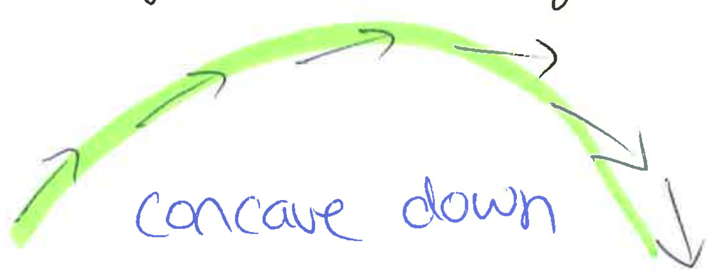
What if $f'' > 0$?

slope is increasing



What if $f'' < 0$?

slope is decreasing



When f'' changes sign we have an **inflection point**: where we change from concave up to concave down.

NOTE: This means at $x = c,\ f''(c) = 0$ or could be undefined but f'' must go from positive to negative.

Example: Find the inflection points and intervals of concavity of the function:

$$f(x) = x^3 - 3x^2 + 9$$

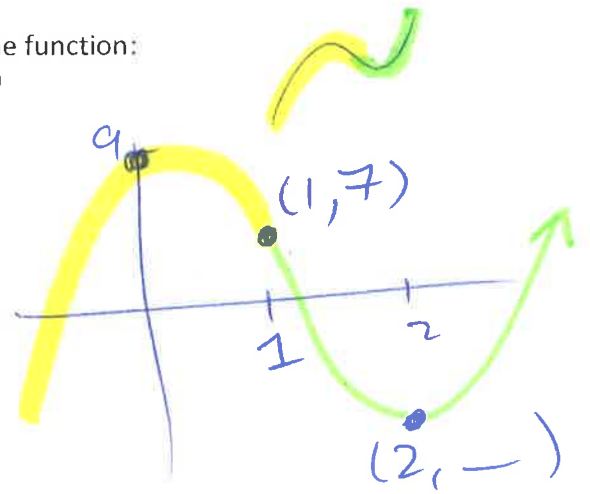
$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f''(x) = 6x - 6 = 0$

$\rightarrow x = 1$

+ f'' concave up $x > 1$

concave down $x < 1$



Practice: Find the inflection points and intervals of concavity of the function:

$$g(x) = x^4 - 24x^2 + 10x$$

/ $y = x^4$

$g'(x) = 4x^3 - 48x + 10$

$g''(x) = 12x^2 - 48 = 0$

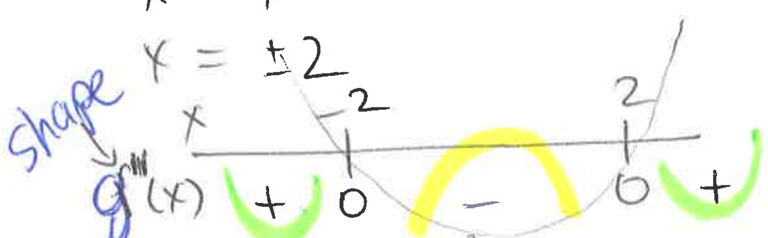
$x^2 = 4$

$x = \pm 2$

concave up when $|x| > 2$

$x > 2$ or $x < -2$

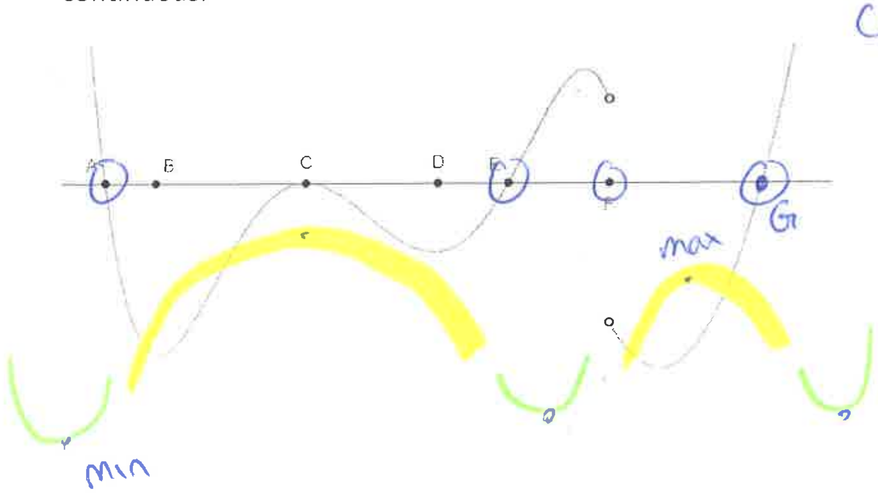
concave down $-2 < x < 2$



Practice: Given the graph of $\frac{d^2h}{dx^2}$, where are the inflection points and where is h concave up? Assume h is continuous.

concave up when

$x < A,\ E < x < F,\ x > G$



$h(x)$

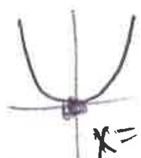
We note that when $f'' > 0$ we could have a minimum 

And if $f'' < 0$ then we could have a maximum 

This builds the **Second Derivative Test!** If we know that $f'(c) = 0$ then we could have a

- If $f''(c) > 0$ then we have a local min
- If $f''(c) < 0$ then we have a local max
- If $f''(c) = 0$ then we have

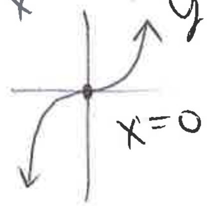
$y = x^4 \quad y'' = 12x^2$



min

$x=0$

$y = x^3 \quad y'' = 6x$



$x=0$

no max/min

inconclusive!

Example: Find the extrema of the following function using second derivative test.

$$k(x) = x^4 + 4x^3 - 7$$

$k'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$

possible max/min @ $x = 0, -3$

$k''(x) = 12x^2 + 24x$

$k''(0) = 0$:(

use first deriv. test

$k''(-3) = 36 > 0$ 

local min @ $x = -3$

Practice: Find the extrema of the following function using second derivative test.

x	$x < A$	A	$A < x < B$	B	$B < x < C$	undefined	$x > C$
$m''(x)$	Positive	0	Negative	undefined	Positive	undef	Positive
$m'(x)$	0 For some $a < A$	5	0 For some $a_b \in (A, B)$	undefined	0 For some $b_c \in (B, C)$	undefined	0 For some $c > C$

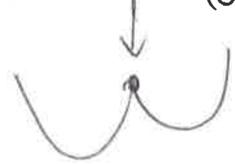
@ $x = a$ local min

@ $x = a_b$ local max

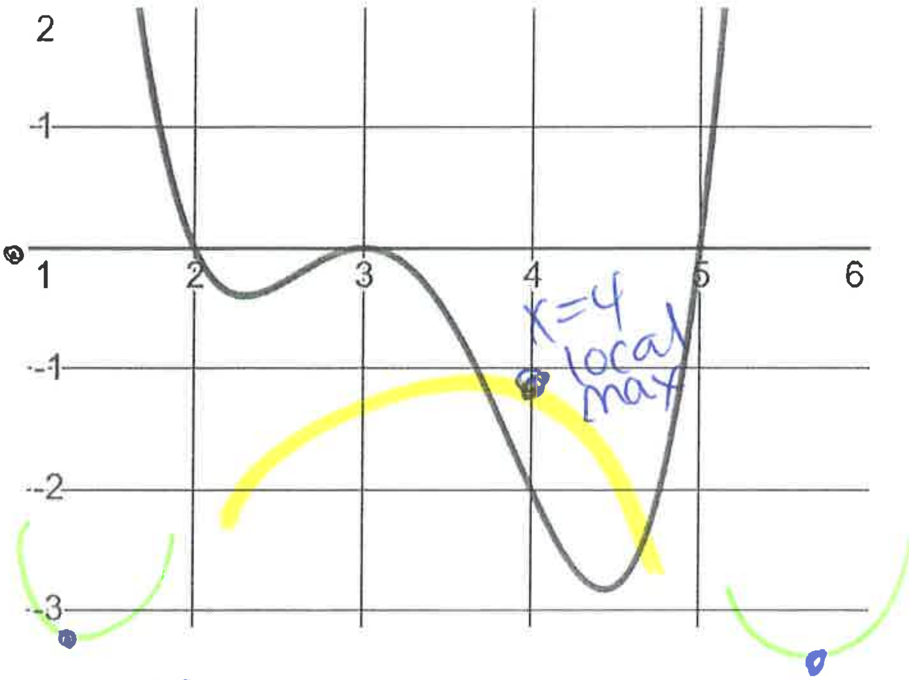
@ $x = b_c$ local min

local max @ $x = C$

@ $x = c$ local min



Practice: Consider the graph of n'' below. If $n'(x) = 0$ when $x = 0.7, 4,$ and 5.3 then determine where the extrema of n occur and the type of extrema.



x=0.7
local
min

x=5.3
local min

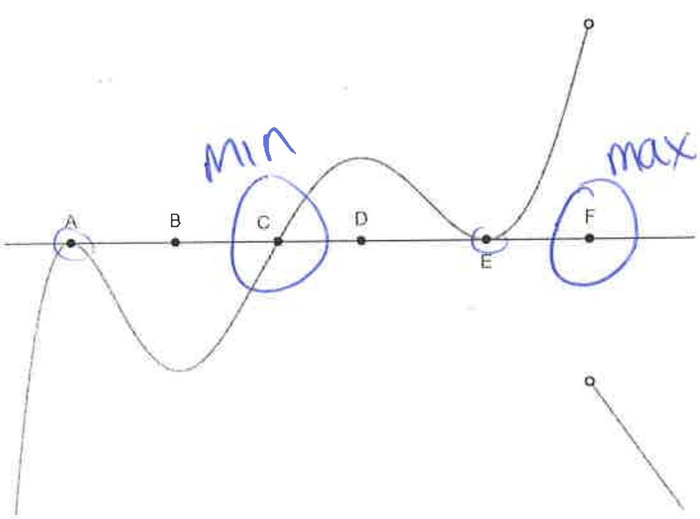
one of these
is the absolute

Practice Problems: 5.3: # 1, 2a-f, 3-9
5.4: # 1 (what you need), 3abc

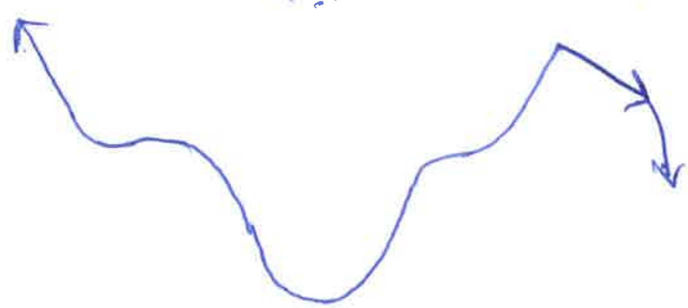
Second Derivative Test and Concavity

Goal:
• Can determine the intervals of concavity of a function f by analyzing the function algebraically, or by analyzing the values of f'' given a graph or table. • Can determine the type of an extremum at a point using the second derivative test and concavity.
Terminology: • Concavity (concave up and down) • Inflection Point • Second Derivative Test
Reminder: • Quiz on Tuesday on Concavity and Slant/Horizontal Asymptotes

Review: Given the following graph of $\frac{df}{dx}$, determine the extrema of f and if they are a max or min.



x		A		C		E		F	
$f'(x)$	−	0	−	0	+	0	+	?	−



Now that we know what the first derivative tells us (namely the slope and if the curve is going up or down). What does the second derivative tell us?

slope of the slope ⇒ how much the slope changes

What if $f'' > 0$?

slope gets more positive

concave up

What if $f'' < 0$?

slope gets more negative

concave down

When f'' changes sign we have an **inflection point**: where we change from concave up to concave down.

NOTE: This means at $x = c$, $f''(c) = 0$ or could be undefined but f'' must go from positive to negative.

Example: Find the inflection points and intervals of concavity of the function:

$$f(x) = x^3 - 3x^2 + 9$$ – curve

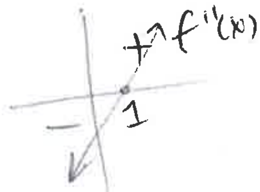
$f'(x) = 3x^2 - 6x \rightarrow$ slope

$f''(x) = 6x - 6 \rightarrow$ shape

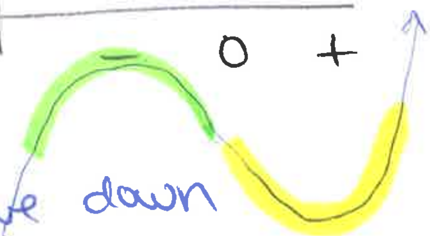
$= 0$

$\Rightarrow 6x - 6 = 0$

$x = 1$



x		1	
$f''(x)$	–	0	+



concave down $x < 1$

concave up $x > 1$

Practice: Find the inflection points and intervals of concavity of the function:

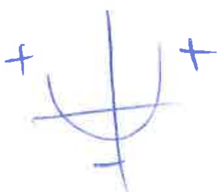
$$g(x) = x^4 - 24x^2 + 10x$$

$g'(x) = 4x^3 - 48x \rightarrow$ slope

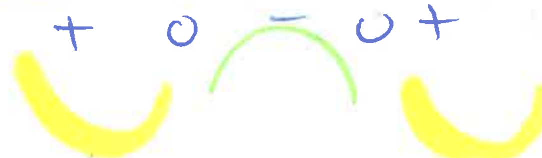
$g''(x) = 12x^2 - 48 \rightarrow$ shape

$= 0$

$\Rightarrow x = \pm 2$



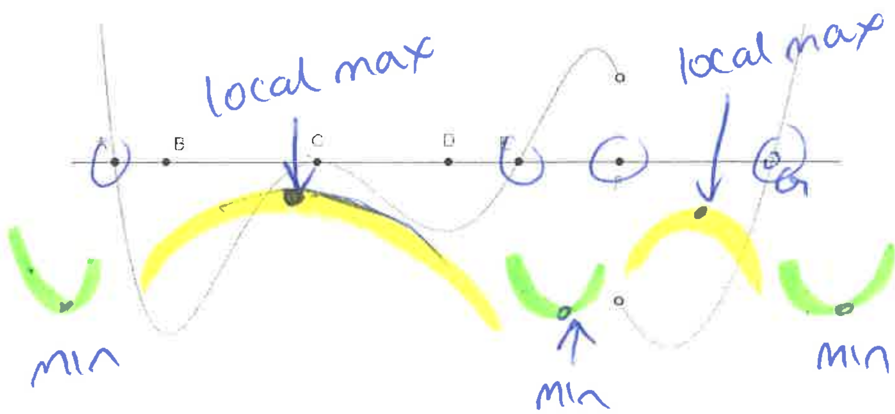
x		-2		2	
$g''(x)$	+	0	–	0	+



concave up when $x < -2$ or $x > 2 \equiv |x| > 2$

concave down $-2 < x < 2$

Practice: Given the graph of $\frac{d^2h}{dx^2}$, where are the inflection points and where is h concave up? Assume h is continuous.



concave up

$x < A$, $E < x < F$, $x > G$

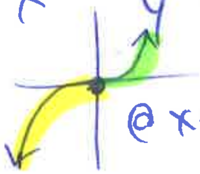
We note that when $f'' > 0$ we could have a minimum 

And if $f'' < 0$ then we could have a maximum 

This builds the **Second Derivative Test!** If we know that $f'(c) = 0$ then we could have a

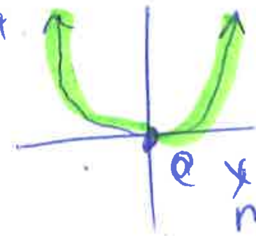
- If $f''(c) > 0$ then we have local min
- If $f''(c) < 0$ then we have local max
- If $f''(c) = 0$ then we have

$y = x^3$ $y'' = 6x$



@ $x=0$ neither

$y = x^4$



$y'' = 12x^2 \geq 0$

@ $x=0$ min

Example: Find the extrema of the following function using second derivative test.

$$k(x) = x^4 + 4x^3 - 7$$

$k'(x) = 4x^3 + 12x^2 = 0 = 4x^2(x+3)$ @ $x = 0, -3$

$k''(x) = 12x^2 + 24x$

$k''(0) = 0$:(

1st Deriv. Test.

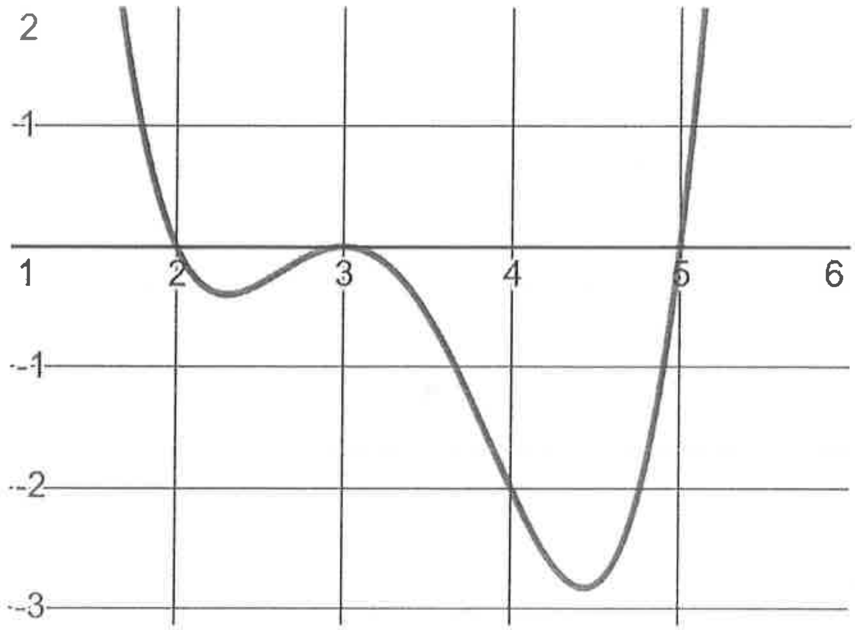
$k''(-3) = 36 > 0$

local min.

Practice: Find the extrema of the following function using second derivative test.

x	$x < A$	A	$A < x < B$	B	$B < x < C$	C	$x > C$
$m''(x)$	Positive	0	Negative	undefined	Positive	undef.	Positive
$m'(x)$	0 For some $a < A$	5	0 For some $a_b \in (A, B)$	undefined	0 For some $b_c \in (B, C)$	~~0~~ undef.	0 For some $c > C$

Practice: Consider the graph of n'' below. If $n'(x) = 0$ when $x = 0.7, 4$, and 5.3 then determine where the extrema of n occur and the type of extrema.



Practice Problems: 5.3: # 1, 2a-f, 3-9 5.4: # 1 (what you need), 3abc