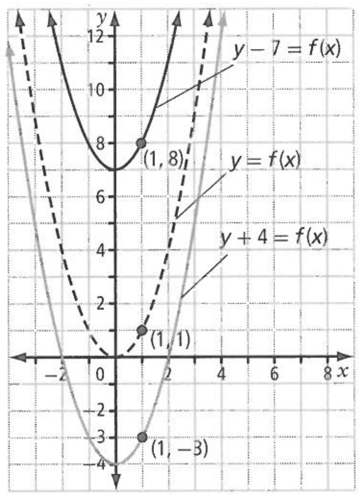
Chapter 1 Function Transformations

1.1 Horizontal and Vertical Translations

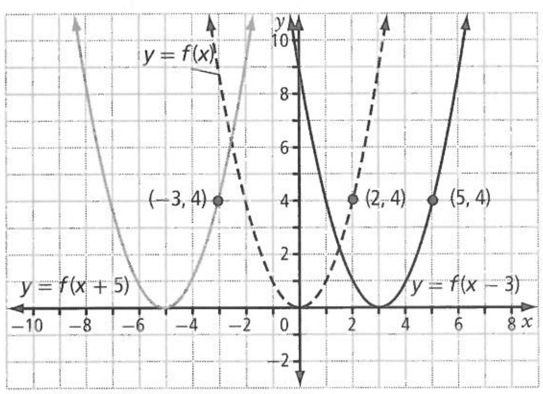
KEY IDEAS

- A translation can move the graph of a function up or down (vertical translation) and right or left (horizontal translation). A translation moves each point on the graph by the same fixed amount so that the location of the graph changes but its shape and orientation remain the same.

- A vertical translation of function $y = f(x)$ by k units is written $y - k = f(x)$. Each point (x, y) on the graph of the base function is mapped to $(x, y + k)$ on the transformed function. Note that the sign of k is opposite to the sign in the equation of the function.
  - If k is *positive*, the graph of the function moves *up*.
    Example: In $y - 7 = f(x)$, $k = 7$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x, y + 7)$. If $f(x) = x^2$, as illustrated, (1, 1) maps to (1, 8).
  - If k is *negative*, the graph of the function moves *down*.
    Example: In $y + 4 = f(x)$, $k = -4$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x, y - 4)$. If $f(x) = x^2$, (1, 1) maps to (1, −3).



- A horizontal translation of function $y = f(x)$ by h units is written $y = f(x - h)$. Each point (x, y) on the graph of the base function is mapped to $(x + h, y)$ on the transformed function. Note that the sign of h is opposite to the sign in the equation of the function.
  - If h is *positive*, the graph of the function shifts to the *right*.
    Example: In $y = f(x - 3)$, $h = 3$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x + 3, y)$. If $f(x) = x^2$, (2, 4) maps to (5, 4).



  - If h is *negative*, the graph of the function shifts to the *left*.
    Example: In $y = f(x + 5)$, $h = -5$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x - 5, y)$. If $f(x) = x^2$, (2, 4) maps to (−3, 4).

- Vertical and horizontal translations may be combined. The graph of $y - k = f(x - h)$ maps each point (x, y) in the base function to $(x + h, y + k)$ in the transformed function.

Working Example 1: Graph Translations of the Form $y - k = f(x - h)$

a) For $f(x) = |x|$, graph $y + 6 = f(x - 4)$ and give the equation of the transformed function.

b) For $f(x)$ as shown, graph $y + 5 = f(x + 2)$.

Solution

a) For $f(x) = |x|$, the transformed function $y + 6 = f(x - 4)$ is represented by $y + 6 = |x - 4|$.

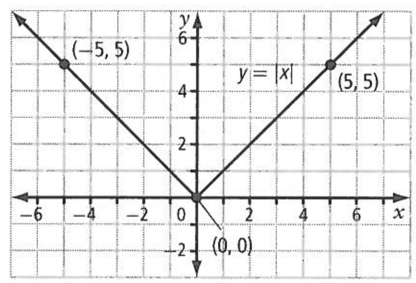
h = ________ means a horizontal translation ________ units to the ________.
(*left* or *right*)

k = ________ means a vertical translation ________ units ________.
(*up* or *down*)

The parameters h and k have the opposite signs to what appear in the equation.

Key points: (x, y) maps to $(x + h, y + k)$

(x, y)	→	$(x + h, y + k)$
(−5, 5)	→	
(0, 0)	→	
(5, 5)	→	



Add these points to your graph and draw in the lines. Be sure to continue the lines to the edge of the graph.

b) The function $y = f(x)$ shown in the graph below will be transformed as follows:
$y + 5 = f(x + 2)$.

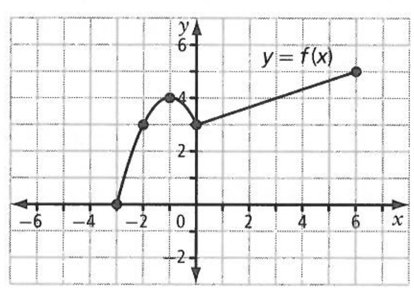
The translated function should be congruent to the base function.

h = ________ means a horizontal translation ________ units to the ________.
(*left* or *right*)

k = ________ means a vertical translation ________ units ________.
(*up* or *down*)

Choose key points from the graph (e.g., maximum and minimum values, endpoints) and map them to new coordinates under the transformation. Then, graph the new function.

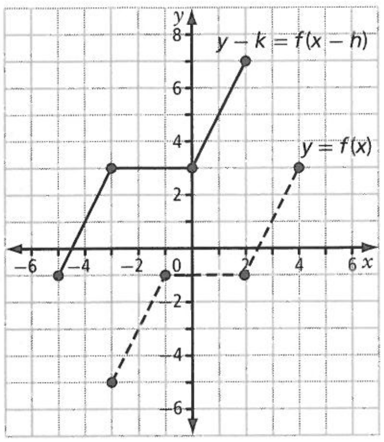
(x, y)	→	$(x + h, y + k)$
(−3, 0)	→	



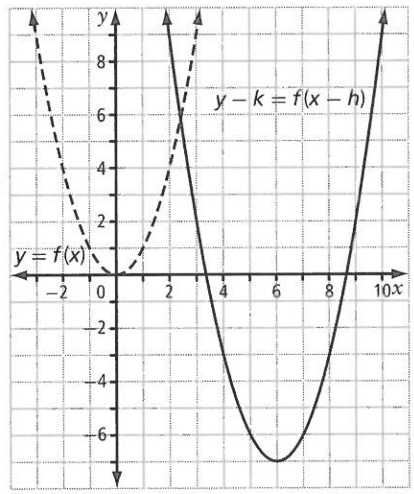
Working Example 2: Determine the Equation of a Translated Function

Determine an equation of the form $y - k = f(x - h)$ given the following graphs of $f(x)$ and of the transformed function.

a)



b)



Solution

a) Verify that the shapes are congruent by comparing slopes and lengths of line segments.

Identify key points in the base function and where they are mapped to in the translation.

Work backward from the graph to determine the parameters h and k.

$(x, y) \rightarrow$ ________________

$h =$ ________ $k =$ ________

This function is not easily described with an equation, so continue to call the base function $y = f(x)$. The equation describing the transformed function is:

b) Verify that the shapes are congruent by looking at the step pattern, starting at the vertex.

Identify key points (e.g., maximum and minimum values, intercepts).

$(x, y) \rightarrow$ ________________

$h =$ ________ $k =$ ________

What is the equation of the base function? (Hint: What kind of function is it?)

What is the equation of the transformed function?

Adding k to both sides of the equation $y - k = (x - h)^2$ will give the equation of a parabola in vertex form. Verify that this works.

Also see Example 3 on pages 10 and 11 of *Pre-Calculus 12*.

Check Your Understanding

Practise

1. Identify the values of the parameters h and k for each of the following functions.

   a) $y = f(x - 10)$ $h =$ _______ $k =$ _______

   b) $y - 3 = f(x + 2)$ $h =$ _______ $k =$ _______

   c) $y = f(x - 17) + 13$ $h =$ _______ $k =$ _______

   d) $y + 7 = (x + 1)^2$ $h =$ _______ $k =$ _______

   e) $y - 4 = |x|$ $h =$ _______ $k =$ _______

You may need to rearrange the equation before answering.

2. Given $h = 2$ and $k = -5$, write an equation for each transformed function $y - k = f(x - h)$.

   a) $f(x) = x^2$

   b) $f(x) = |x|$

   c) $f(x) = \frac{1}{x}$

3. Describe, using mapping notation, how the graphs of the following functions can be obtained from the graph of $y = f(x)$. Then, describe each transformation in words.

   a) $y = f(x - 25)$ $(x, y) \rightarrow$ _______________

   This represents a _______________ translation _______________ by _______ units.
   (*horizontal* or *vertical*) (*right*/*left*/*up*/*down*)

   b) $y + 50 = f(x)$ $(x, y) \rightarrow$ _______________

   This represents a _______________ translation _______________ by _______ units.
   (*horizontal* or *vertical*) (*right*/*left*/*up*/*down*)

   c) $y - 10 = f(x + 20)$ $(x, y) \rightarrow$ _______________

   This represents a ____.

See also #8 on page 13 of *Pre-Calculus 12*.

4. Given the graph of $y = f(x)$, graph the transformed function on the same set of axes. Write the transformation using mapping notation.

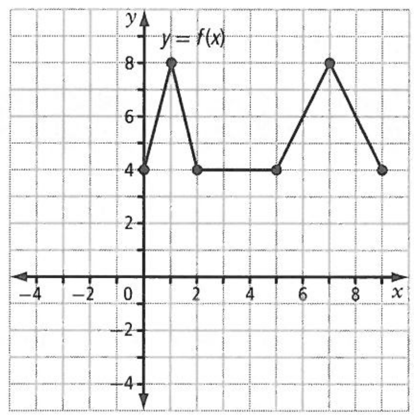
a) Graph $y + 7 = f(x + 2)$.

$h =$ ________ means a horizontal translation ________ units to the ________.
(*left* or *right*)

$k =$ ________ means a vertical translation ________ units ________.
(*up* or *down*)

Key points: (x, y) maps to $(x + h, y + k)$

(x, y)	$\rightarrow$	$(x + h, y + k)$
(0, 4)	$\rightarrow$	

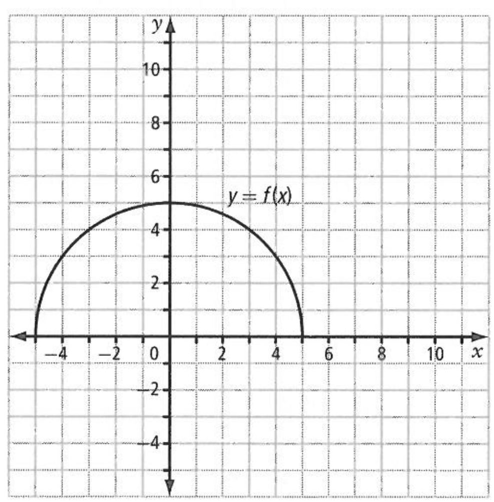


Verify that your mapping is correct by checking that the translated function is congruent to the base.

b) Graph $y + 2 = f(x - 5)$.

Key points:

(x, y)	$\rightarrow$	$(x + h, y + k)$



Apply

5. The graph of the function $f(x) = x^2$ is translated 6 units to the right and 4 units down to form the transformed function $y = g(x)$.

   a) Identify the values of the parameters h and k. $h =$ ______ $k =$ ______

   b) Write the transformation $f(x) \rightarrow g(x)$ using mapping notation.

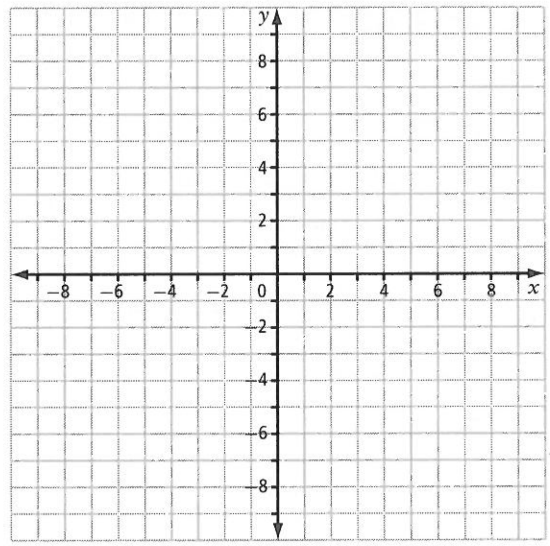
   ____

   c) Determine the equation of the function $y = g(x)$. ______________________

   d) Graph $f(x)$ and $g(x)$ on the same set of axes.

   Key points:

(x, y)	$\rightarrow$	$(x + h, y + k)$



   e) Compare the vertex of $f(x)$ to that of $g(x)$. What do you notice?

   Vertex of $f(x)$:

   Vertex of $g(x)$:

   f) Compare the domain and range of $f(x)$ to those of $g(x)$. What do you notice?

   Domain of $f(x)$:

   Domain of $g(x)$:

   Range of $f(x)$:

   Range of $g(x)$:

6. The graph of the function $f(x) = \frac{1}{x}$ is translated 5 units to the left and 2 units up to form the transformed function $y = g(x)$.

a) Identify the values of the parameters h and k. $h =$ ________ $k =$ ________

b) Write the transformation $f(x) \rightarrow g(x)$ using mapping notation.

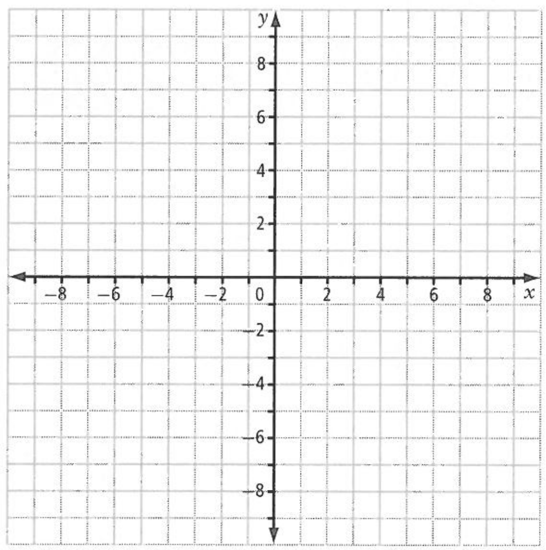
____

c) Determine the equation of the function $y = g(x)$. ____________________________

d) Graph $f(x)$ and $g(x)$ on the same set of axes.

Key points:

(x, y)	$\rightarrow$	$(x + h, y + k)$



e) Compare the domain, range, and asymptotes of $f(x)$ to those of $g(x)$. What do you notice?

Domain of $f(x)$:

Domain of $g(x)$:

Range of $f(x)$:

Range of $g(x)$:

Horizontal asymptote of $f(x)$:

Horizontal asymptote of $g(x)$:

Vertical asymptote of $f(x)$:

Vertical asymptote of $g(x)$:

Connect

7. Complete the table using equations, mapping notation, and diagrams. Be sure to include information on the location of key features (such as vertex and asymptotes) where applicable.

Function	Horizontal Translation		Vertical Translation	
	to the right 1 unit	to the left 3 units	up 2 units	down 4 units
Quadratic $y = x^2$				
Absolute value $y = \|x\|$				
Reciprocal $y = \frac{1}{x}$				
Any function $y = f(x)$				