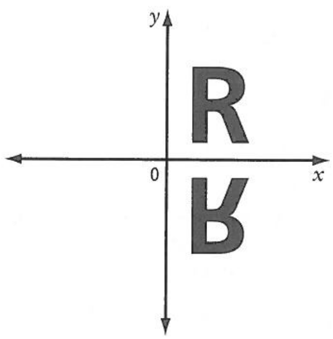
1.2 Reflections and Stretches

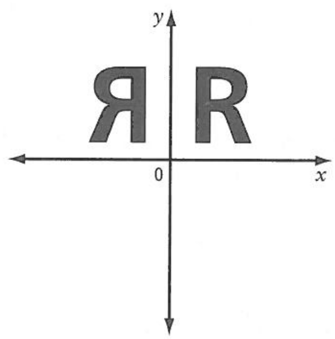
KEY IDEAS

- A reflection creates a mirror image of the graph of a function across a line of reflection. Any points where the function crosses the line of reflection do not move (invariant points). A reflection may change the orientation of the function but its shape remains the same.

Vertical reflection:
- $y = -f(x)$
- $(x, y) \rightarrow (x, -y)$
- line of reflection: x-axis
- also known as a reflection in the x-axis

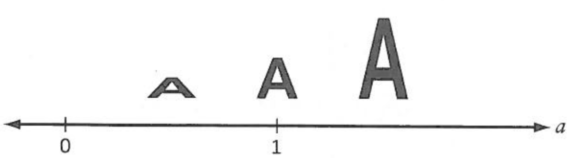


Horizontal reflection:
- $y = f(-x)$
- $(x, y) \rightarrow (-x, y)$
- line of reflection: y-axis
- also known as a reflection in the y-axis

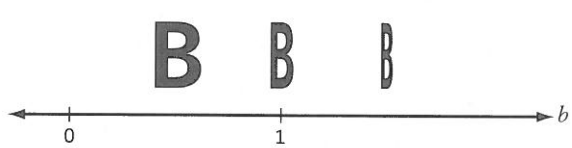


- A stretch changes the shape of a graph but not its orientation. A vertical stretch makes a function shorter (compression) or taller (expansion) because the stretch multiplies or divides each y-coordinate by a constant factor while leaving the x-coordinate unchanged. A horizontal stretch makes a function narrower (compression) or wider (expansion) because the stretch multiplies or divides each x-coordinate by a constant factor while leaving the y-coordinate unchanged.

Vertical stretch by a factor of $|a|$:
- $y = af(x)$ or $\frac{1}{a}y = f(x)$
- $(x, y) \rightarrow (x, ay)$
- shorter: $0 < |a| < 1$
- taller: $|a| > 1$



Horizontal stretch by a factor of $\frac{1}{|b|}$:
- $y = f(bx)$
- $(x, y) \rightarrow \left(\frac{1}{b}x, y\right)$
- wider: $0 < |b| < 1$
- narrower: $|b| > 1$



Working Example 1: Graph Reflections of a Function $y = f(x)$

Given $y = f(x)$, graph the indicated transformation on the same set of axes. Give the mapping notation representing the transformation. Identify any invariant points.

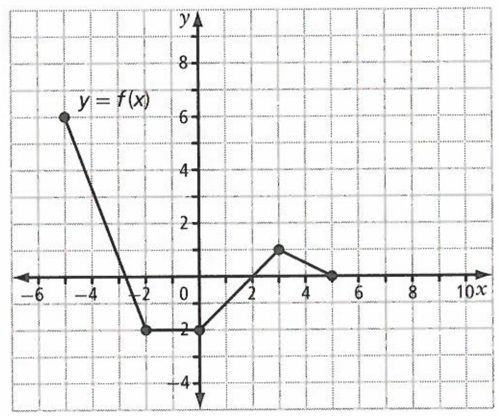
a) $y = f(-x)$ **b)** $y = -f(x)$

Solution

a) $y = f(-x)$ represents a ________________ reflection of the function in the ________ -axis.
(*horizontal* or *vertical*) (x or y)

Key points:

(x, y)	$\rightarrow$	
$(-6, 6)$	$\rightarrow$	
$(-2, -2)$	$\rightarrow$	
$(0, -2)$	$\rightarrow$	
$(3, 1)$	$\rightarrow$	
$(5, 0)$	$\rightarrow$	

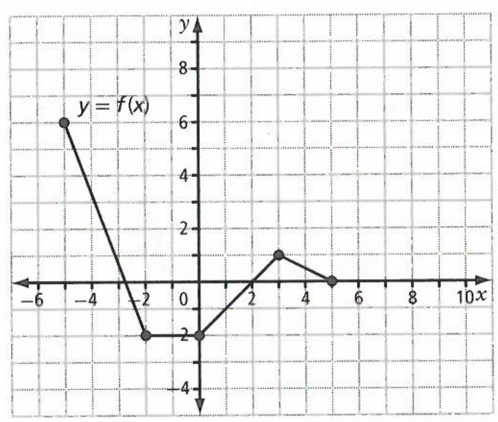


Invariant point(s): ________________

b) $y = -f(x)$ represents a ________________ reflection of the function in the ________ -axis.
(*horizontal* or *vertical*) (x or y)

Key points:

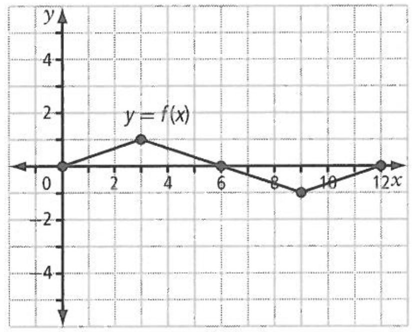
(x, y)	$\rightarrow$	
$(-6, 6)$	$\rightarrow$	
$(-2, -2)$	$\rightarrow$	
$(0, -2)$	$\rightarrow$	
$(3, 1)$	$\rightarrow$	
$(5, 0)$	$\rightarrow$	



Invariant point(s): ________________

Working Example 2: Graph Vertical and Horizontal Stretches of a Function $y = f(x)$

Given $y = f(x)$, graph $y = 5f(3x)$ on the same set of axes. Give the mapping notation representing the transformation.



Solution

For $y = 5f(3x)$,

$a =$ ________ represents a vertical stretch by a factor of ________.
Will the new graph be shorter or taller than the graph of the base function?

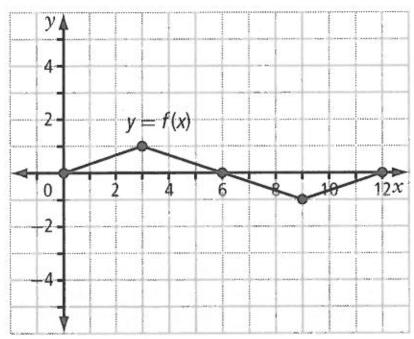
$b =$ ________ represents a horizontal stretch by a factor of ________.
Will the new graph be wider or narrower than the graph of the base function?

Apply the transformations in two stages: vertical stretch first, followed by the horizontal stretch. Graph using key points at the end of each stage. Use a different colour for each stage.

Does the order matter?

Vertical stretch by a factor of 5, followed by a horizontal stretch by a factor of $\frac{1}{3}$:

(x, y)	→		→	
(0, 0)				
(3, 1)				

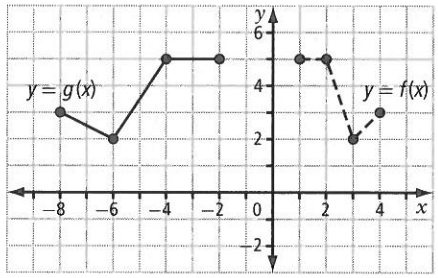


Compare the domain and range of the base function to those of the image. How many patterns (width-wise) could fit in the width of the base function?

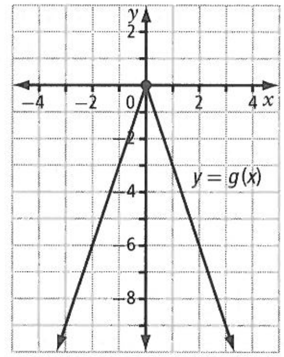
Working Example 3: Write the Equation of a Transformed Function

The graph of the function $y = f(x)$ has been transformed by a series of stretches and/or reflections. Write the equation of the transformed function $g(x)$.

a)



b)



Solution

a) Key points:

$f(x)$	→	$g(x)$
(1, 5)	→	(−2, 5)
(2, 5)	→	
(3, 2)	→	
(4, 3)	→	
(x, y)	→	

Has the orientation changed (reflection)?

In which direction?

Has the shape changed (stretch)?

In which direction?

By how much?

Equation: ______________________

b) The base function $f(x)$ is not shown. What must it be? Add it to the graph.

Key points:

$f(x)$	→	$g(x)$
(−3, 3)	→	
(0, 0)	→	
(3, 3)	→	
(x, y)	→	

Has the orientation changed (reflection)?

In which direction?

Has the shape changed (stretch)?

In which direction?

By how much?

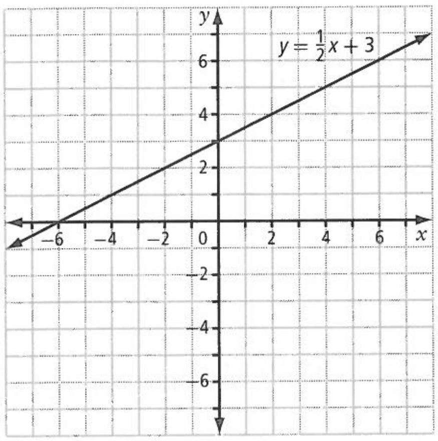
Equation: ______________________

How can you tell whether $g(x)$ is narrower or taller than $f(x)$? Does it matter? What other common function has this property?

Check Your Understanding

Practise

1. Graph the horizontal reflection (reflection in the y-axis) of each function. State the equation of the reflected function in simplified form. Note any features of the function that change and any that stay the same.

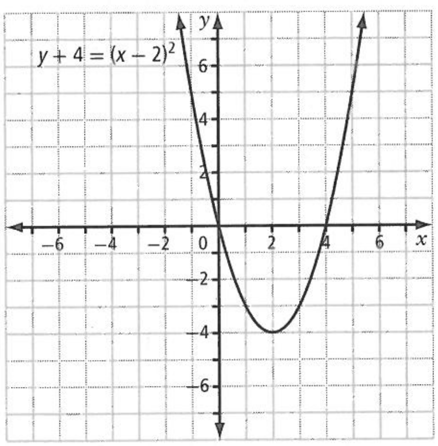
a) 

Equation of function: $y = \frac{1}{2}x + 3$

Equation of reflected function:

____

Notes:

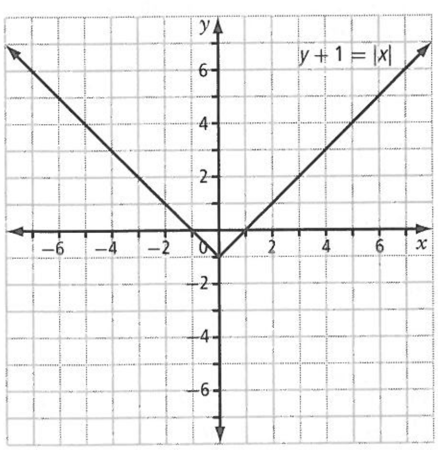
b) 

Equation of function: $y + 4 = (x - 2)^2$

Equation of reflected function:

____

Notes:

c) 

Equation of function: $y + 1 = |x|$

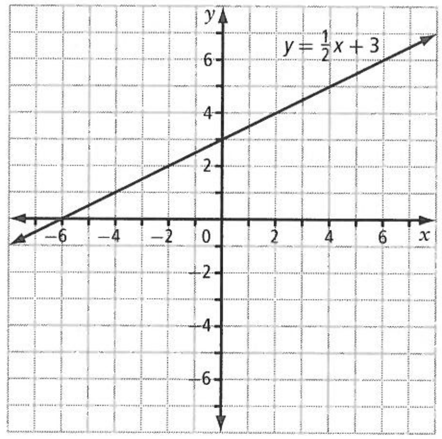
Equation of reflected function:

____

Notes:

2. Graph the vertical reflection (reflection in the x-axis) of each function. State the equation of the reflected function in simplified form. Note any features of the function that change and any that stay the same.

a)



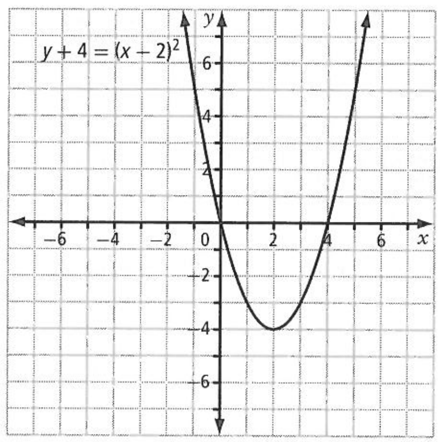
Equation of function: $y = \frac{1}{2}x + 3$

Equation of reflected function:

____

Notes:

b)



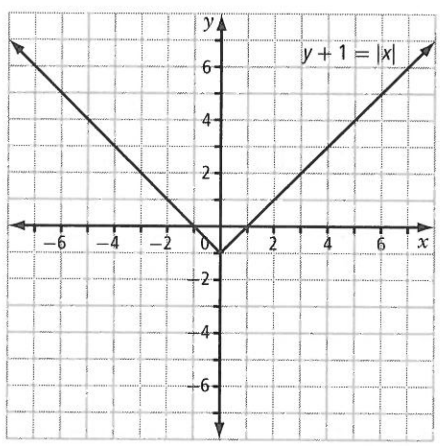
Equation of function: $y + 4 = (x - 2)^2$

Equation of reflected function:

____

Notes:

c)



Equation of function: $y + 1 = |x|$

Equation of reflected function:

____

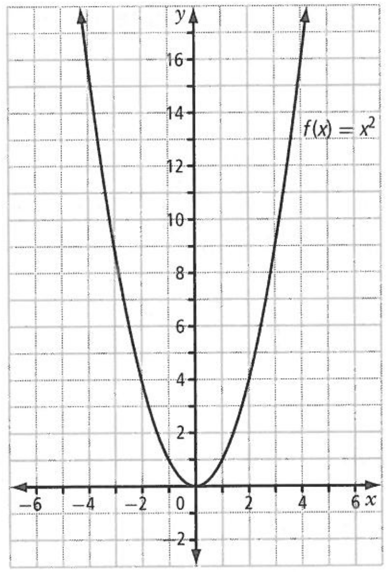
Notes:

3. Given $f(x) = x^2$, graph the following transformations. Give the equation and mapping notation for each transformation.

a) vertical stretch by a factor of $\frac{1}{4}$

Key points: (x, y) maps to (x, ay)

(x, y)	$\rightarrow$	
$(0, 0)$	$\rightarrow$	
$(\pm1, 1)$	$\rightarrow$	
$(\pm2, 4)$	$\rightarrow$	
$(\pm3, 9)$	$\rightarrow$	
$(\pm4, 16)$	$\rightarrow$	

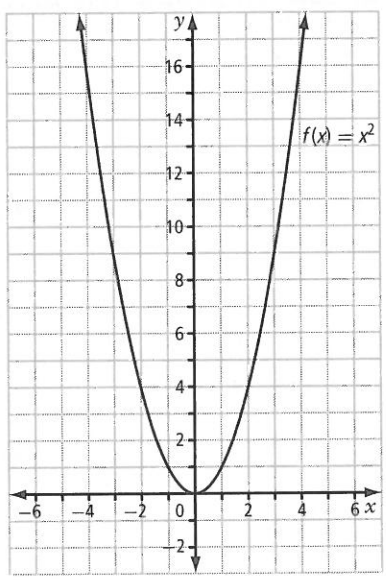


Equation: ____________________

b) horizontal stretch by a factor of 2 (b = reciprocal of the stretch factor)

Key points: (x, y) maps to $\left(\frac{1}{b}x, y\right)$

(x, y)	$\rightarrow$	
$(0, 0)$	$\rightarrow$	
$(\pm1, 1)$	$\rightarrow$	
$(\pm2, 4)$	$\rightarrow$	
$(\pm3, 9)$	$\rightarrow$	
$(\pm4, 16)$	$\rightarrow$	



Equation: ____________________

4. Compare your answers in parts a) and b) of #3.

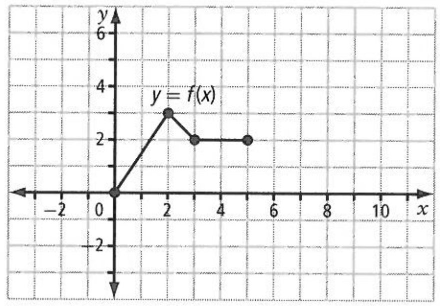
a) Show algebraically why both transformations result in the same transformed function.

b) Give another example of a pair of horizontal and vertical stretches that would result in the same transformed function.

Apply

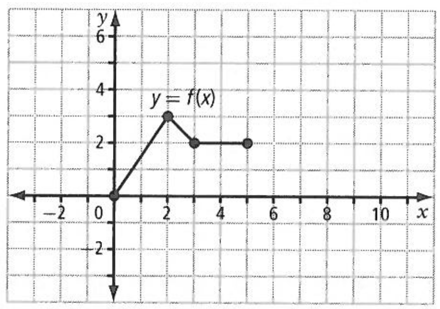
5. Write an equation representing each of the following transformations of $y = f(x)$. Then, graph each transformation.

a) vertical stretch by a factor of 2



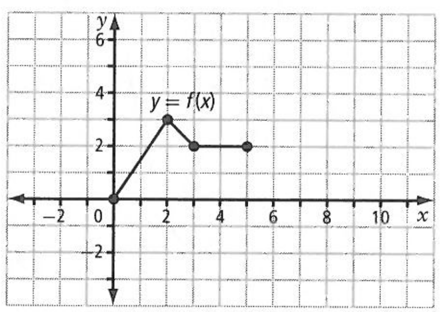
Equation of transformed function:

b) reflection in the x-axis and horizontal stretch by a factor of 2



Equation of transformed function:

c) reflection in the y-axis and horizontal stretch by a factor of $\frac{1}{2}$



Equation of transformed function:

Recall that $y = f(bx)$ results in a horizontal stretch of $\frac{1}{|b|}$.

Connect

6. Use sketches, graphs, equations, mapping notation, and words to describe how to achieve the following transformations to the graph of $y = x^2$.

Shorter	Taller

Narrower	Wider