1.3 Combining Transformations

KEY IDEAS

- Types of transformations include stretches, reflections, and translations.
- Multiple transformations can be applied to the same function. The same order of operations followed when you work with numbers (sometimes called BEDMAS) applies to transformations: first multiplication and division (stretches, reflections), and then addition/subtraction (translations).

  $y - k = af(b(x - h))$

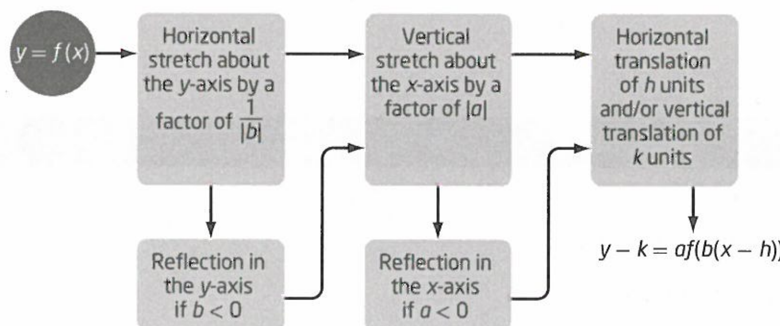
- The following three-step process will help you to keep organized.

  Step 1: horizontal stretch by a factor of $\frac{1}{|b|}$ followed by reflection in the y-axis if $b < 0$

  Step 2: vertical stretch by a factor of $|a|$ followed by reflection in the x-axis if $a < 0$

  Step 3: horizontal and/or vertical translations (h and k)

$$(x, y) \rightarrow \left(\frac{1}{b}x, y\right) \rightarrow \left(\frac{1}{b}x, ay\right) \rightarrow \left(\frac{1}{b}x + h, ay + k\right)$$



Working Example 1: Combinations of Transformations

Graph each of the following transformed functions. Show each stage of the transformation in a different colour and label each stage.

a) $y + 2 = -\left|\frac{1}{3}x - \frac{4}{3}\right|$ **b)** $y - 5 = \frac{1}{2}f(-x)$

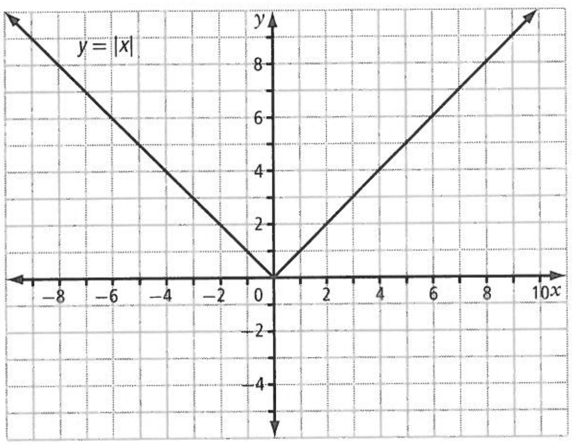
Solution

a) The transformations applied to $y = |x|$ to obtain $y + 2 = -\left|\frac{1}{3}x - \frac{4}{3}\right|$ are, in order,

i) ____________________

ii) reflection in the x-axis

iii) ____________________



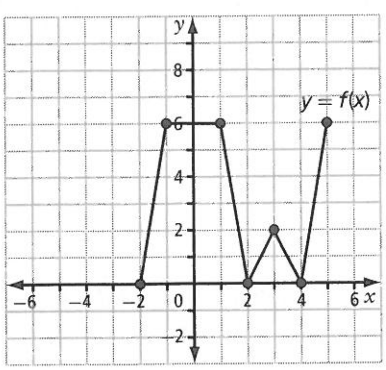
Be sure to factor the input of the function so that it is in the form $f[b(x - h)]$.

b) The transformations applied to $y = f(x)$ to obtain $y - 5 = \frac{1}{2}f(-x)$ are, in order,

i) reflection in the y-axis

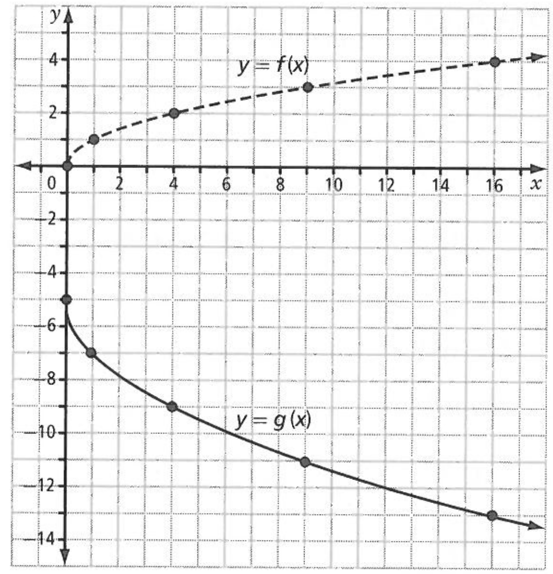
ii) vertical stretch by a factor of ________

iii) ____________________



Working Example 2: Determine the Equation of a Translated Function

Determine an equation for $g(x)$ of the form $y - k = af(b(x - h))$ given the graphs of $y = f(x)$ and of the transformed function $y = g(x)$.



Solution

Work backward through the three stages of transformations.

Horizontal and vertical translations:
Any points $(x, 0)$ on the graph of $f(x)$ are unaffected by vertical stretches and reflections.

Any points $(0, y)$ on the graph of $f(x)$ are unaffected by horizontal stretches and reflections.

So, the key point $(0, 0)$ on the graph of $f(x)$ can be used to determine the horizontal and vertical translations.

In the equation for $g(x)$, $h =$ ________ and $k =$ ________.

Before proceeding, add a sketch to the graph above that shows $g(x)$ without translations.

Vertical stretches and reflections:
- Is the transformed function reflected across the x-axis? (Y/N)
- Is the transformed function the same shape as the base function? (Y/N)

If the answer to the second question is no, measure the vertical distance between the x-axis and key points on $f(x)$. Then, measure the vertical distance between the x-axis and the image points on $g(x)$ and subtract the value of k. Compare the vertical distances.

In $g(x)$, $a =$ ________.

Horizontal stretches and reflections:
- Is the transformed function reflected in the y-axis? (Y/N)
- Is the transformed function the same shape as the base function? (Y/N)

If the answer to the second question is no, measure the horizontal distance between the y-axis and key points on $f(x)$. Then, measure the horizontal distance between the y-axis and the image points on $g(x)$ and subtract the value of h. Compare the horizontal distances.

In $g(x)$, $b =$ ________.

Now, put the transformations together.

Equation representing $g(x)$: ____________________

Also see Example 3 on pages 37 and 38 of *Pre-Calculus 12*.

Check Your Understanding

Practise

You may need to factor the equation before answering.

1. Describe, in order, the transformations represented by each equation.

a) $y + 5 = 4f(-x)$

i)

ii)

iii)

b) $y = -f(2x + 14)$

i)

ii)

iii)

c) $y = 1.75f[0.25(x - 1.5)]$

i)

ii)

iii)

d) $y - 3 = -\frac{1}{2}f(-3x - 3)$

i)

ii)

iii)

2. Determine the equation of each transformed function.

a) $y = f(x)$ is stretched horizontally by a factor of 6, reflected in the x-axis, and translated 7 units down.

b) $y = |x|$ is reflected in the y-axis, stretched vertically by a factor of $\frac{1}{2}$, and translated 3 units to the right.

c) $y = x^2$ is reflected in the x-axis, stretched horizontally by a factor of 3, and translated so that the vertex is at $(10, -4)$.

3. The key point $(1, 10)$ is on the graph of $y = f(x)$. Determine the coordinates of its image point under each transformation.

a) $y + 4 = f(x - 5)$

$(x, y) \rightarrow$

$(1, 10) \rightarrow$

b) $y = -f(x + 12)$

$(x, y) \rightarrow$

$(1, 10) \rightarrow$

c) $y = 3f(-0.5x + 10)$

$(x, y) \rightarrow$

$(1, 10) \rightarrow$

This is similar to #6 on page 39 of *Pre-Calculus 12*.

4. If the key point $(-2, -8)$ is on the graph of $y = f(x)$, determine the coordinates of its image point under each of the transformations in #3.

Apply

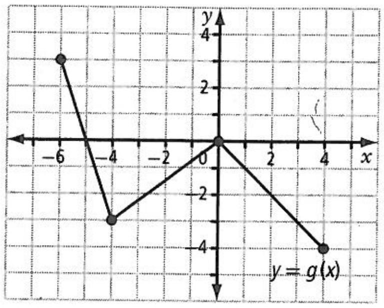
5. The graph of the function $y = g(x)$ is given. Graph each of the following transformations of the function. Show each stage of the transformation in a different colour.

a) $y + 2 = -g(2x)$

i)

ii)

iii)

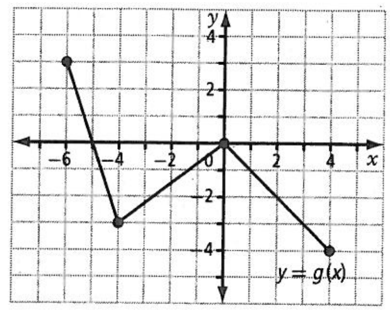


b) $y = g(-4x + 12)$

i)

ii)

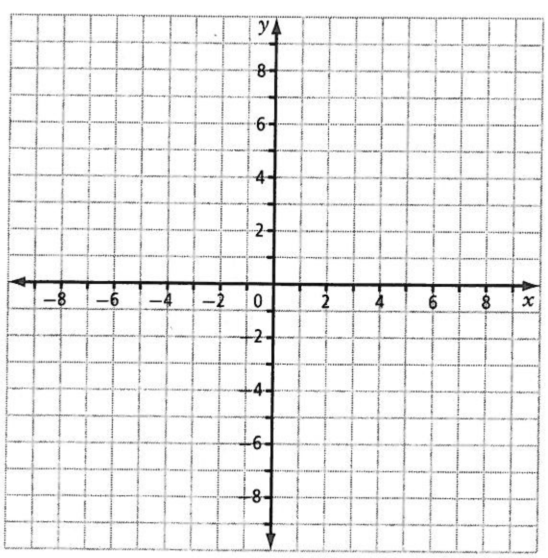
iii)



6. The graph of the function $f(x) = |x|$ is stretched vertically by a factor of 2, reflected in the x-axis, and translated 6 units to the left and 3 units down to form the transformed function $y = g(x)$.

a) Determine the equation of the function $y = g(x)$.

b) Graph $y = g(x)$.

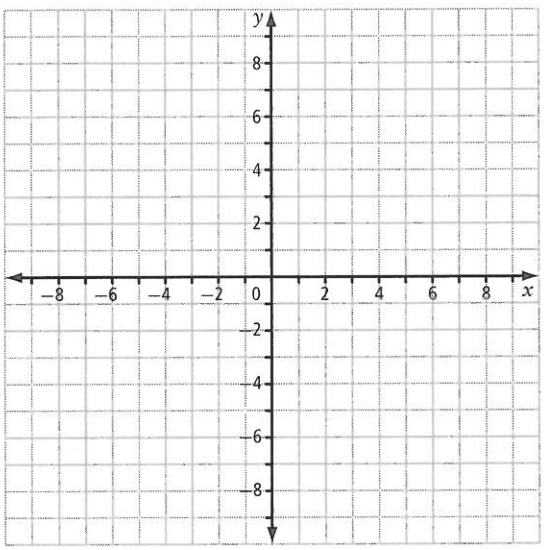


Start by graphing the base function $y = f(x)$.

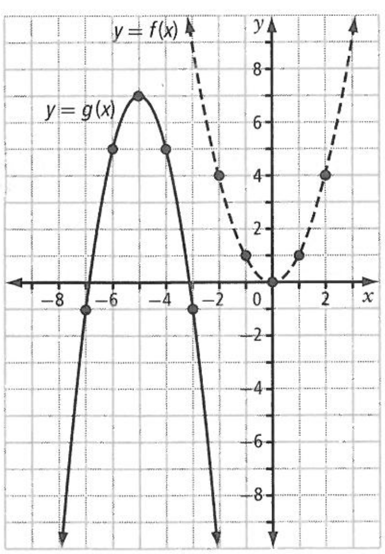
7. The graph of the function $f(x) = \frac{1}{x}$ is stretched horizontally by a factor of 4, reflected in the x-axis, and translated 4 units to the right and 1 unit down to form the transformed function $y = g(x)$.

a) Determine the equation of the function $y = g(x)$.

b) Graph $y = g(x)$.

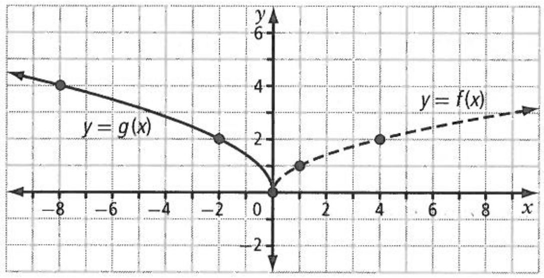


8. Determine an equation for $g(x)$ of the form $y - k = af(b(x - h))$ given the graphs of $y = f(x)$ and the transformed function $y = g(x)$.



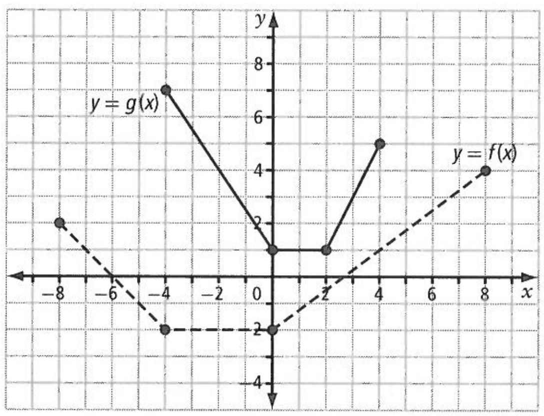
Equation:

9. Determine an equation for $g(x)$ of the form $y - k = af(b(x - h))$ given the graphs of $y = f(x)$ and the transformed function $y = g(x)$.



Equation:

10. Determine an equation of the form $y - k = af(b(x - h))$ given the following graphs of $y = f(x)$ and of the transformed function $y = g(x)$.



Consider each of the possible types of transformations in reverse order: translations, vertical stretches and reflections, and horizontal stretches and reflections.

For additional similar questions, see #10 on page 40 of *Pre-Calculus 12*.

Connect

11. Choose your transformations:

a) horizontal stretch by a factor of ________ (a = ________); horizontal reflection? (Y/N)

b) vertical stretch by a factor of ________ (b = ________); vertical reflection?(Y/N)

c) translations by ____

Write equations representing the transformed function after each stage of the transformation. Simplify each equation if necessary.

Function	Horizontal Stretch and/or Reflection	Vertical Stretch and/or Reflection	Translations
$y = f(x)$			
$y = x$			
$y = \|x\|$			
$y = x^2$			
$y = \frac{1}{x}$			