1.4 Inverse of a Relation

KEY IDEAS

- The inverse of a function $y = f(x)$ is denoted $y = f^{-1}(x)$ if the inverse is a function. The -1 is not an exponent because f represents a function, not a variable. You have already seen this notation with trigonometric functions. Example: $\sin^{-1}(\theta)$, where $f(\theta) = \sin(\theta)$ and the variable is θ.
- The inverse of a function reverses the processes represented by that function. For example, the process of squaring a number is reversed by taking the square root. The process of taking the reciprocal of a number is reversed by taking the reciprocal again.
- To determine the inverse of a function, interchange the x- and y-coordinates.

$$(x, y) \rightarrow (y, x)$$

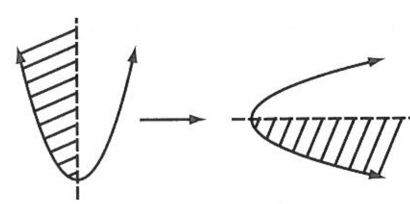
or

$$y = f(x) \rightarrow x = f(y)$$

or

reflect in the line $y = x$

- When working with an equation of a function $y = f(x)$, interchange x for y. Then, solve for y to get an equation for the inverse. If the inverse is a function, then $y = f^{-1}(x)$.
- If the inverse of a function is not a function (recall the vertical line test), restrict the domain of the base function so that the inverse becomes a function. You will see this frequently with quadratic functions. For example, the inverse of $f(x) = x^2$, $x \geq 0$, is $f^{-1}(x) = \sqrt{x}$. The inverse will be a function only if the domain of the base function is restricted.
- Restricting the domain is necessary for any function that changes direction (increasing to decreasing, or vice versa) at some point in the domain of the function.



Working Example 1: Determine the Inverse of a Relation

Determine the inverse of the given relation when it is described as

a) an equation **b)** a table of values **c)** a graph

Solution

a) $f(x) = -\frac{2}{3}x + 5$

Steps:

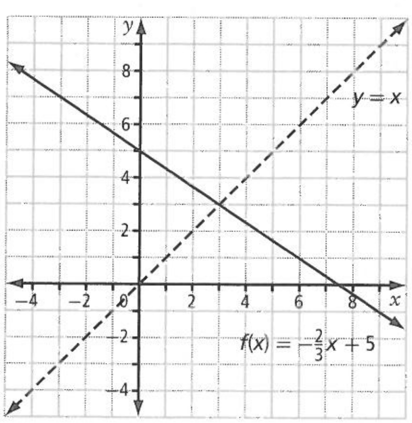
1. Substitute y for $f(x)$. $y = -\frac{2}{3}x + 5$
2. Interchange x and y. ______ $= -\frac{2}{3}$______ $+ 5$
3. Solve for y.

4. If the inverse is a function, substitute $f^{-1}(x)$ for y.

b) Key points: x- and y-coordinates are interchanged

(x, y)	→	(y, x)
$(-3, 7)$	→	
$(0, 5)$	→	
$(3, 3)$	→	
$(6, 1)$	→	

c) The inverse of $f(x)$ is the reflection in the line $y = x$. Choose key points and interchange the x- and y-coordinates.



Identify any invariant point(s): ______

Note that the equation, table of values, and graph all represent the same function. It is a good idea when working with inverses to verify your algebra using a graph.

What happens to the y-intercept?
What happens to the x-intercept?

Is the inverse of a linear function always a function?

Working Example 2: Determine the Equation of the Inverse of a Quadratic Function

Determine algebraically the equation of the inverse of the function $f(x) = (x + 3)^2 - 1$. Verify graphically that the relations are inverses of each other.

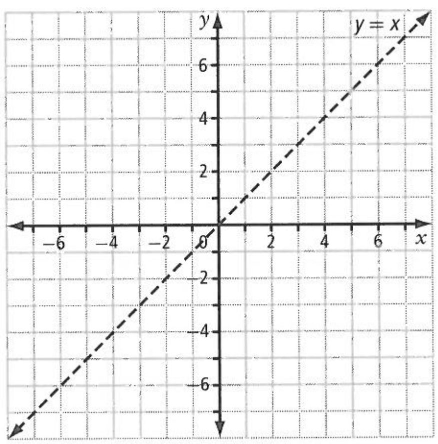
Solution

Let $y = f(x)$. To determine the inverse, interchange x and y (everywhere it says x, write y, and vice versa).

Solve for y.

When taking the square root of both sides, be sure to take the square root of the whole expression on each side.

Next, create the graph of $f(x) = (x + 3)^2 - 1$ and its inverse. Compare the graph to your solution above to verify that your algebra is correct (compare, for example, coordinates of the vertex and direction of opening).



To graph the inverse, choose some key points from the base function and interchange the x- and y-coordinates.

Is the inverse a function? (Y/N)

Divide the inverse into two branches (+ and −) at the vertex. Do the same for the base function.

In the base function, the equation of the axis of symmetry is

$x =$ ________

Restrict the domain to $\{x \mid x \geq$ ________, $x \in \mathrm{R}\}$.

Restricting the domain of $f(x)$ to the positive branch of the original parabola ($x \geq -3$) gives only the positive root from the equation of the inverse relation $y = \pm\sqrt{x + 1} - 3$.

Therefore, for the function $f(x) = (x + 3)^2 - 1$, $x \geq -3$, the inverse is

$f^{-1}(x) =$ ____________________.

Alternatively, you could choose the negative branch and write that the inverse of the function:

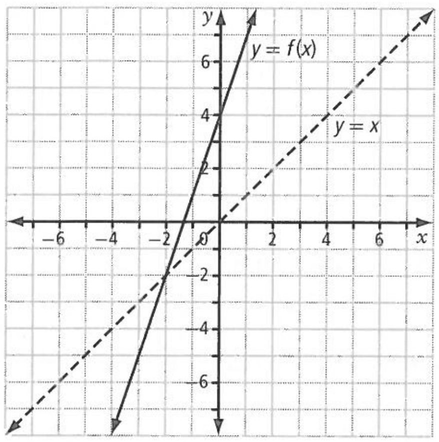
$f(x) = (x + 3)^2 - 1$, $x \leq -3$, is $f^{-1}(x) =$ ____________________.

Check Your Understanding

Practise

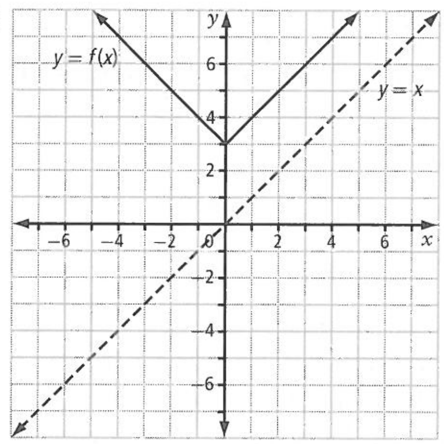
1. Graph the inverse relation of each function below. Determine whether the inverse is a function. Identify any invariant points.

a)



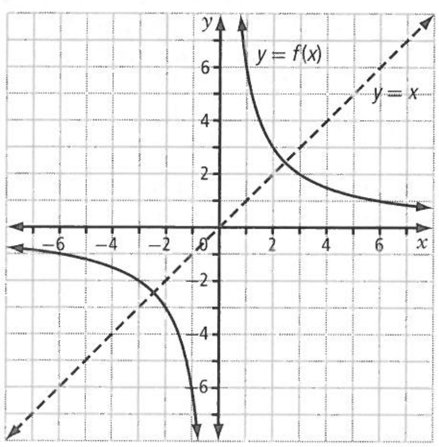
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

b)



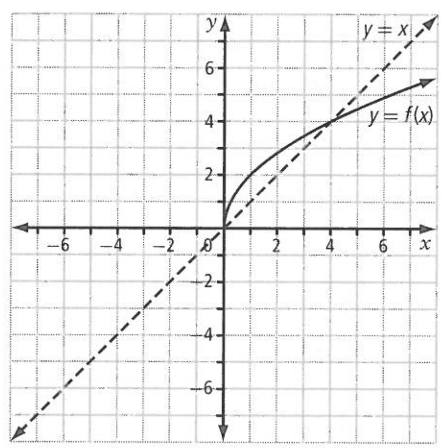
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

c)



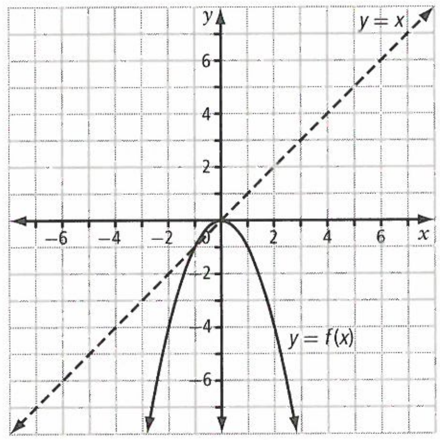
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

d)



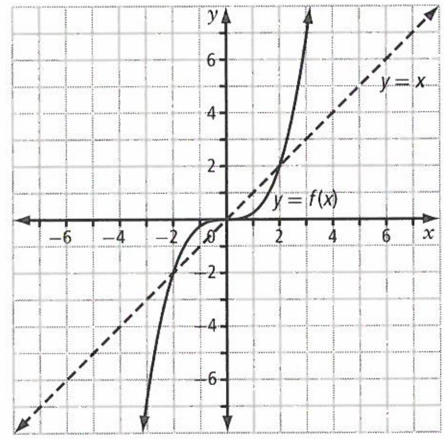
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

e)



The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

f)



The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

2. Determine algebraically the inverse of each function. Verify by sketching the graph of the function and its inverse.

a) $f(x) = x - 4$

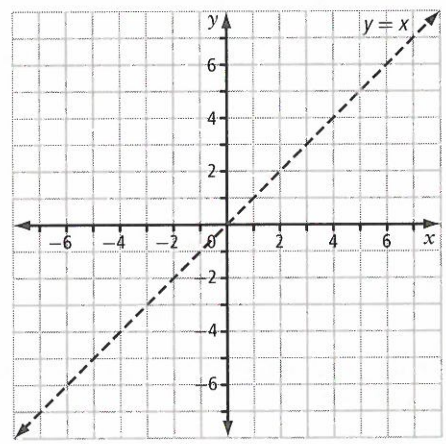
Steps:

1. Substitute y for $f(x)$.

2. Interchange x and y.

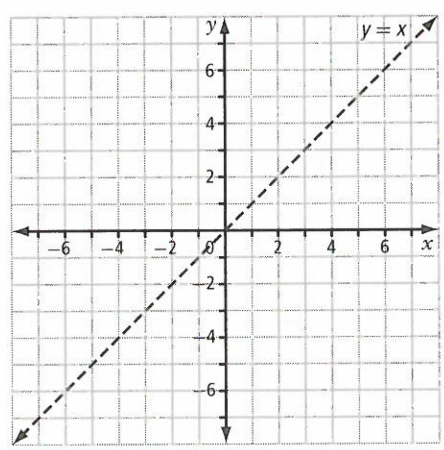
3. Solve for y.

4. Restrict the domain if necessary. Then, substitute $f^{-1}(x)$ for y.



The inverse of $f(x) = x - 4$ is $f^{-1}(x) =$ ________________.

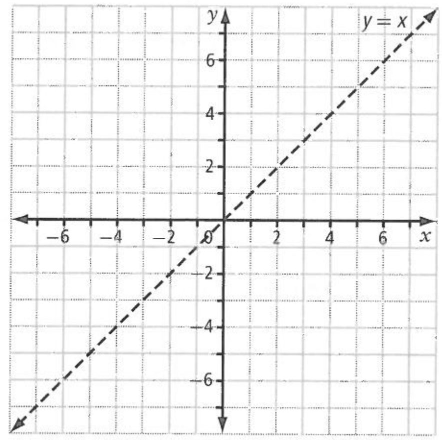
b) $f(x) = -6x - 2$



The inverse of $f(x) = -6x - 2$ is

$f^{-1}(x) =$ ________________.

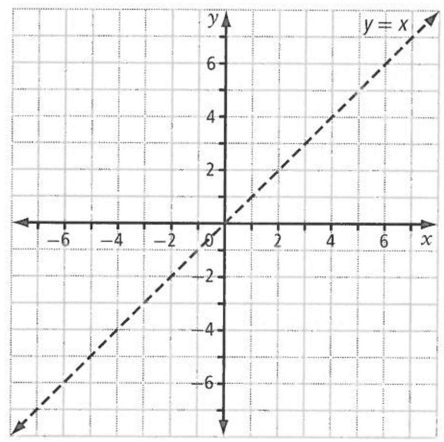
c) $f(x) = \frac{3}{5}x - 3$



The inverse of $f(x) = \frac{3}{5}x - 3$ is

$f^{-1}(x) =$ ________________.

d) $f(x) = \frac{1}{2}(x + 6)$

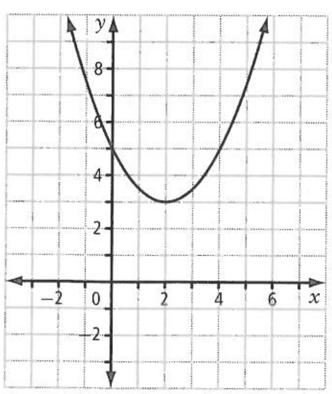


The inverse of $f(x) = \frac{1}{2}(x + 6)$ is

$f^{-1}(x) =$ ________________.

3. For each graph, identify a restricted domain for which the function has an inverse that is also a function.

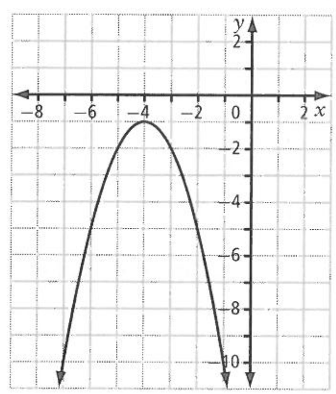
a)



Axis of symmetry: ________________

Domain: ________________________

b)

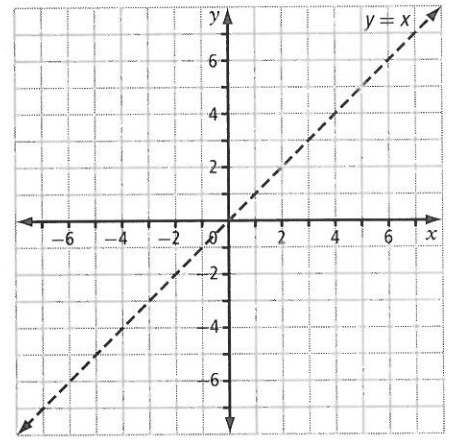


Axis of symmetry: ________________

Domain: ________________________

4. Determine algebraically the inverse of each function. Restrict the domain of the base function so that the inverse is a function. Verify by sketching the graph of the function and its inverse.

a) $f(x) = -x^2 + 6$



Steps:

1. Substitute y for $f(x)$.

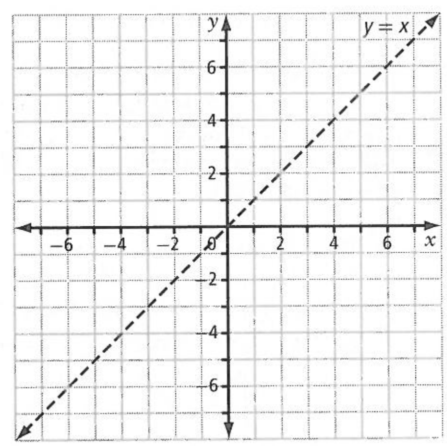
2. Interchange x and y.

3. Solve for y.

4. Restrict the domain if necessary. Then, substitute $f^{-1}(x)$ for y.

The inverse of $f(x) = -x^2 + 6$, ______________, is $f^{-1}(x) =$ ______________.

b) $f(x) = \frac{1}{2}x^2 + 4$



Steps:

1. Substitute y for $f(x)$.

2. Interchange x and y.

3. Solve for y.

4. Restrict the domain if necessary. Then, substitute $f^{-1}(x)$ for y.

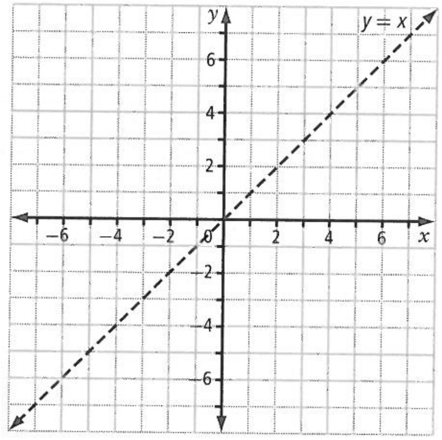
The inverse of $f(x) = \frac{1}{2}x^2 + 4$, ______________, is $f^{-1}(x) =$ ______________.

For more practice with quadratics, try #12 on page 54 of *Pre-Calculus 12*.

Apply

5. Determine the equation of the inverse of $f(x) = x^2 + 6x + 7$. Verify by sketching the graph of the function and its inverse.

Hint: Complete the square.



See also #10 on page 53 of *Pre-Calculus 12*.

6. One of the factors that doctors use to determine the age of a fetus is the crown-to-rump length (CRL) measured during an ultrasound. A recent study determined that the average CRL, in millimetres, of a fetus with gestational age x days could be represented by the function $f(x) = 0.016\,34(x - 26.643)^2$. This formula applies between 6 and 15 weeks of gestation.

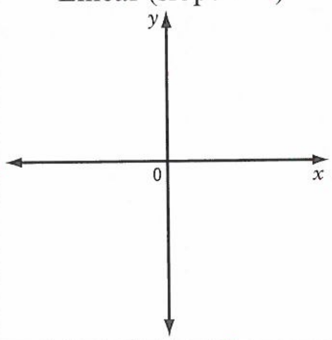
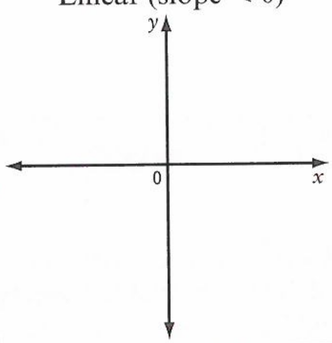
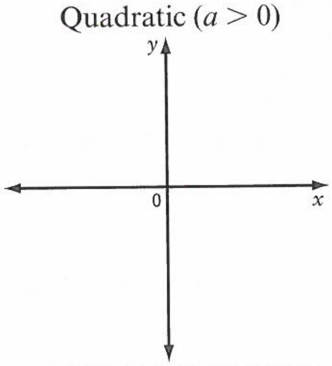
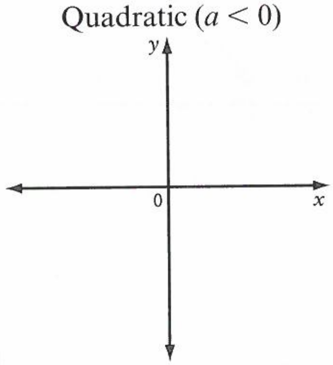
   a) What are the restrictions on the domain of this function?

   b) Determine an equation that would allow a doctor to determine gestational age, in days, if the CRL, in millimetres, is known.

   c) If the CRL of a fetus is 7.4 cm, predict the gestational age in weeks.

Connect

7. Complete the table using words, equations, and diagrams. A few prompts are included to help you get started.

$f(x)$ is …	Key Features of $f(x)$	$f^{-1}(x)$ is …	Key Features of $f^{-1}(x)$
Linear (slope > 0) 	slope: y-intercept: x-intercept:		slope: y-intercept: x-intercept:
Linear (slope < 0) 			
Quadratic ($a > 0$) 	vertex:		vertex:
Quadratic ($a < 0$) 			

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