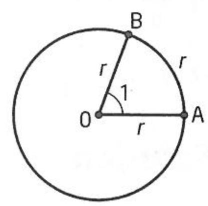
Chapter 4 Trigonometry and the Unit Circle

4.1 Angles and Angle Measure

KEY IDEAS

- One radian is the measure of the central angle subtended in a circle by an arc equal in length to the radius of the circle.

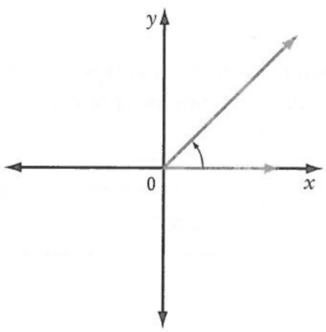


- Travelling one rotation around the circumference of a circle causes the terminal arm to turn $2\pi r$. Since $r = 1$ on the unit circle, $2\pi r$ can be expressed as 2π, or 2π radians.

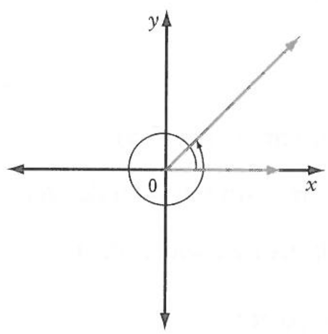
  You can use this information to translate rotations into radian measures. For example,

1 full rotation (360°) is 2π radians	$\frac{1}{6}$ rotation (60°) is $\frac{\pi}{3}$ radians
$\frac{1}{2}$ rotation (180°) is π radians	$\frac{1}{8}$ rotation (45°) is $\frac{\pi}{4}$ radians
$\frac{1}{4}$ rotation (90°) is $\frac{\pi}{2}$ radians	$\frac{1}{12}$ rotation (30°) is $\frac{\pi}{6}$ radians

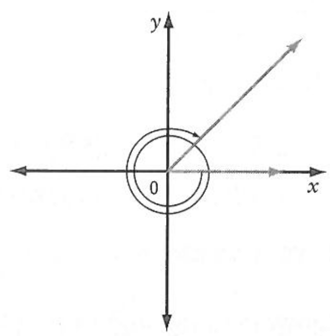
- Angles in standard position with the same terminal arms are coterminal. For an angle in standard position, an infinite number of angles coterminal with it can be determined by adding or subtracting any number of full rotations.
- Counterclockwise rotations are associated with positive angles. Clockwise rotations are associated with negative angles.



quadrant I angle



positive angle > 360°



negative angle

- The general form of a coterminal angle (in degrees) is $\theta \pm 360°n$, where n is a natural number (0, 1, 2, 3, …) and represents the number of revolutions. The general form (in radians) is $\theta \pm 2\pi n$, $n \in \mathrm{N}$.
- Radians are especially useful for describing circular motion. Arc length, a, means the distance travelled along the circumference of a circle of radius r. For a central angle θ, in radians, $a = \theta r$.

Working Example 1: Convert Between Degree and Radian Measure

Draw each angle in standard position. Convert each degree measure to radian measure and each radian measure to degree measure. Give answers as both exact and, if necessary, approximate measures to the nearest hundredth of a unit.

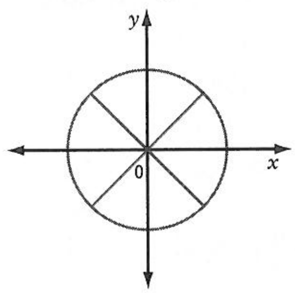
a) 135°

b) $\frac{5\pi}{6}$

c) 4

Solution

a) Draw the angle 135° in standard position. What is its reference angle? ________



The angle 135° is $\frac{3}{4}$ of a half circle, or $\frac{3}{4} \times \pi$.

Convert the degree measure to radian measure.

$360° = 2\pi$

$1° = \frac{\square}{\square}$

$= $ ________

135° = 135 ________

When expressing an angle measure in radians, no unit is necessary.

= ________ (exact measure in terms of π)

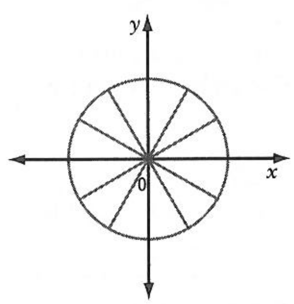
= ________ (approximate measure, to two decimal places)

Use the π button on your calculator.

Is your answer reasonable? Verify using the diagram.

b) Draw the angle $\frac{5\pi}{6}$ in standard position.

Each half circle (π) is divided into 6 segments.



Convert the radian measure to degree measure.

$2\pi =$ ________°

$\pi =$ ________°

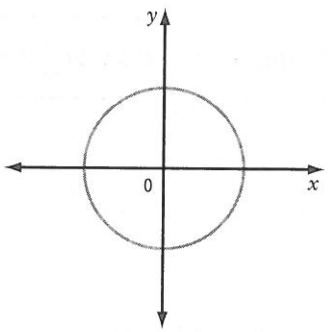
$\frac{5\pi}{6} = \frac{5(\square°)}{6}$

$\frac{5\pi}{6} =$ ________°

When expressing an angle measure in degrees, the degree symbol ° is used.

Is your answer reasonable? Verify using the diagram.

c) The measure 4 has no units shown, so it represents an angle in radians.
Draw the angle 4 in standard position. Hint: $\pi \approx 3.14$ is equivalent to 180°.



Convert the radian measure to degree measure.

2π radians $= 360°$

1 radian $= \frac{\square°}{\square}$

4 radians $=$ ________° (exact measure)

$=$ ________° (approximate measure, to two decimal places)

Is your answer reasonable? Verify using the diagram.

Refer to pages 168–169 of *Pre-Calculus 12* for two other valid procedures for converting between degrees and radians. Use whichever method makes the most sense to you.

Working Example 2: Identify Coterminal Angles and Express Them in General Form

a) Identify the angles coterminal with 210° that satisfy the domain $-720° \leq \theta < 720°$. Express the angles coterminal with 210° in general form.

b) Identify the angles coterminal with $-\frac{3\pi}{4}$ within the domain $-4\pi \leq \theta < 4\pi$. Express angles coterminal with $-\frac{3\pi}{4}$ in general form.

Solution

a) To determine angles coterminal with 210°, add and subtract multiples of 360° (1 full rotation).

The given domain of $0° \pm 720°$ is $\pm$ 2 rotations. Determine the coterminal angles and cross out any that fall outside the given domain.

$\theta - 2(360°)$	$\theta - 360°$	$\theta + 360°$	$\theta + 2(360°)$

The values that satisfy the domain $-720° \leq \theta < 720°$ are ______________.

For n rotations, the general form for angles coterminal with 210° is ______________, $n \in N$.

b) To determine coterminal angles, add and subtract multiples of 2π (1 full rotation).

$2\pi = \frac{\square\pi}{4}$

The given domain of $0 \pm 4\pi$ is $\pm$ ________ rotations.

Determine the coterminal angles and cross out any that fall outside the given domain.

$\theta - 2(2\pi)$	$\theta - 2\pi$	$\theta + 2\pi$	$\theta + 2(2\pi)$

The values that satisfy the domain ________ $\leq \theta <$ ________ are

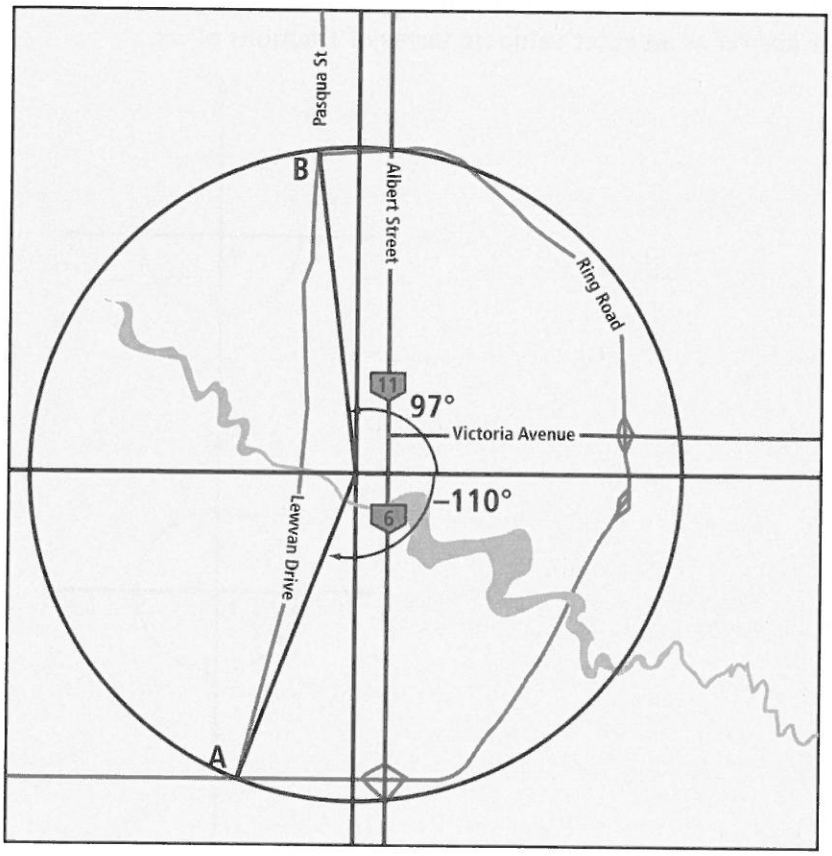
______________.

For n rotations, the general form for angles coterminal with $-\frac{3\pi}{4}$ is ______________, $n \in N$.

See Examples 2 and 3 on pages 170–172 of *Pre-Calculus 12* for more examples.

Working Example 3: Determine Arc Length in a Circle

The ring road around the eastern part of the city of Regina is almost a semicircle. Estimate the length of the ring road (from A to B) if the radius of the circle is 4.9 km.



Solution

Determine the measure of the central angle in radians. Then, use the formula $a = \theta r$ to determine the arc length.

The central (obtuse) angle is ________ (in degrees).

Convert the degree measure to radian measure.

$360° =$ ________

$1° =$ ________

________ $\times 1° =$ ________ $\times \frac{2\pi}{360}$

$\approx$ ____________

Write down a decimal approximation, but keep all digits in your calculator.

For central angles θ expressed in radians, the arc length of a circle of radius r is $a = \theta r$.

Therefore, the length of the ring road is approximately ____________.

What are the units?

The actual distance is 17.6 km. How accurate is your estimate?

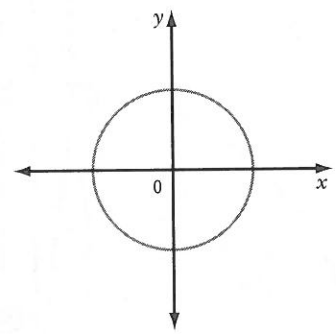
For a similar problem on a larger scale, try #20 on page 178 of *Pre-Calculus 12*.

Check Your Understanding

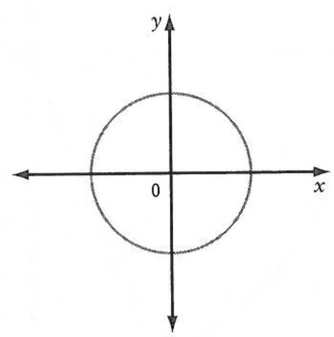
Practise

1. Sketch each angle in standard position. Change each degree measure to radian measure. Express your answer as an exact value (in terms of fractions of π).

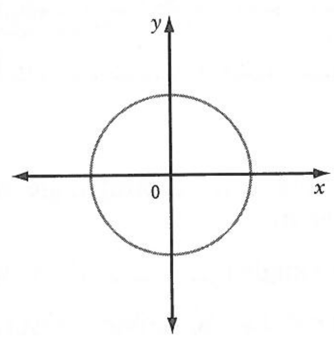
   a) 60°



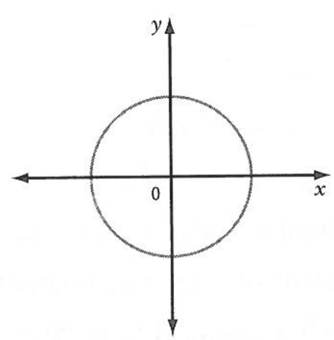
   b) 315°



   c) −210°

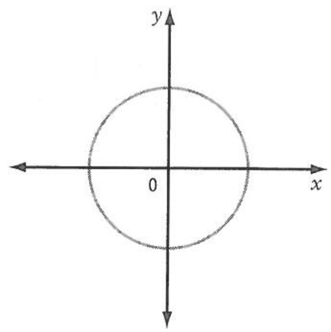


   d) 600°

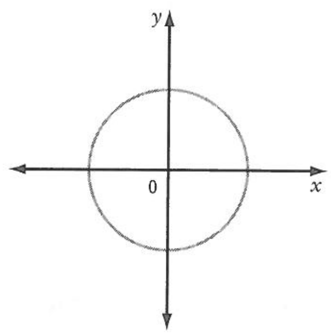


2. Draw each angle in standard position. Change each degree measure to radian measure. Express your answer as a decimal rounded to two decimal places.

a) 101°



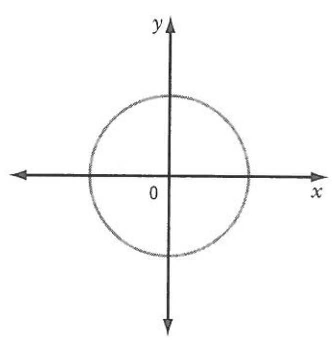
b) 57.3°



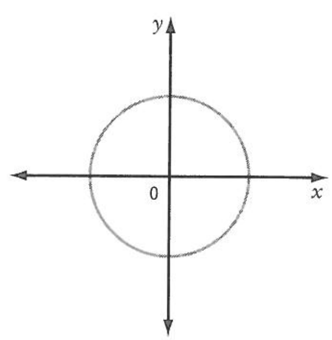
For additional practice, see #3 on page 175 of *Pre-Calculus 12*.

3. Sketch each angle in standard position. Change each radian measure to degree measure. If necessary, round your answer to two decimal places.

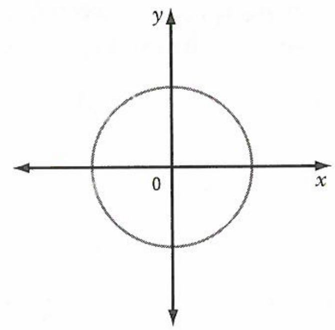
a) $\frac{\pi}{2}$



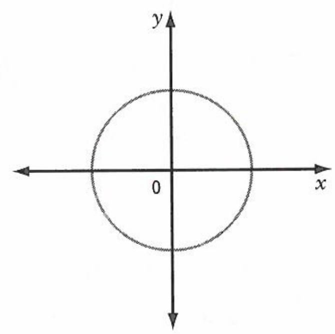
b) $\frac{4\pi}{3}$



c) $-\frac{2\pi}{9}$



d) 2



4. Determine one positive and one negative angle coterminal with each angle given.

a) 349°

b) −487°

c) $\frac{2\pi}{3}$

d) $\frac{9\pi}{4}$

5. For each angle θ, determine all coterminal angles within the given domain. Write an expression for all angles coterminal with θ in general form.

a) $\theta = 255°$ within the domain $-720° \leq \theta < 720°$

$\theta - 2(360°)$	$\theta - 360°$	$\theta + 360°$	$\theta + 2(360°)$

For n rotations, the general form for angles coterminal with 255° is

_______________, $n \in N$.

b) $\theta = \pi$ within the domain $-4\pi \leq \theta < 4\pi$

$\theta - 4\pi$	$\theta - 2\pi$	$\theta + 2\pi$	$\theta + 4\pi$

For n rotations, the general form for angles coterminal with π is

_______________, $n \in N$.

c) $\theta = \frac{5\pi}{6}$ within the domain $-2\pi \leq \theta < 6\pi$

For n rotations, the general form for angles coterminal with $\frac{5\pi}{6}$ is

_______________, $n \in N$.

Also try #11 on page 176 of *Pre-Calculus 12*.

6. Determine the arc length subtended by each central angle. Give answers to the nearest hundredth of a unit.

a) radius 20 cm, central angle $\frac{2\pi}{3}$

b) radius 15 mm, central angle 195°

Apply

7. Angular velocity describes the rate of change in a central angle over time. For example, the change could be expressed in revolutions per minute (rpm), radians per minute, or degrees per second. To determine linear velocity from angular velocity, use the formula $v = \omega r$, where ω is the angular velocity in radians per unit of time and r is the radius of the circular motion.

a) How does the angular velocity formula compare to the formula for arc length?

b) The Great Beijing wheel, a Ferris wheel with a diameter of 198 m, makes 1 revolution in 20 min. What is its angular velocity, in radians per minute? What is the linear velocity of a passenger, in m/s?

c) A bicycle wheel turns at 60 rpm. If the wheels of the bicycle measure 650 mm across, what distance, in metres, does the bicycle travel in 1.00 min?

d) The mean distance from Earth to the moon is 385 000 km. The moon travels around Earth once every 27.2 days. Assuming a circular orbit, what is the linear velocity of the moon, in km/h?

Connect

8. Label the given angles (including the axes) in both degrees and radians, $0° \le \theta < 720°$ and $0 \le \theta < 4\pi$.