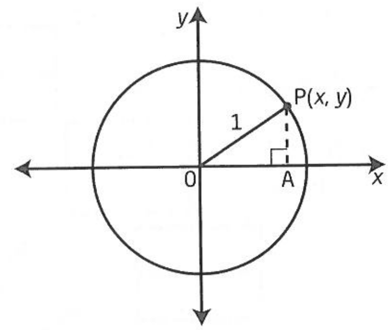
KEY IDEAS

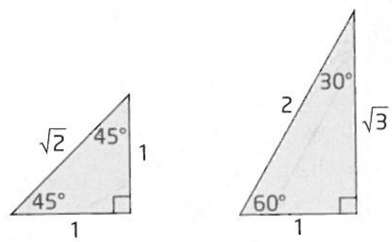
- In general, a circle of radius r centred at the origin has equation $x^2 + y^2 = r^2$.
- The unit circle has radius 1 and is centred at the origin. The equation of the unit circle is $x^2 + y^2 = 1$. All points P(x, y) on the unit circle satisfy this equation.



- An arc length measured along the unit circle equals the measure of the central angle (in radians).

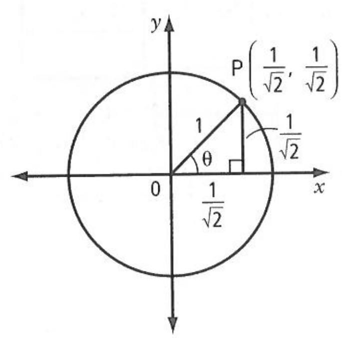
In other words, when $r = 1$, the formula $a = \theta r$ simplifies to $a = \theta$.

- Recall the special right triangles you learned about previously.

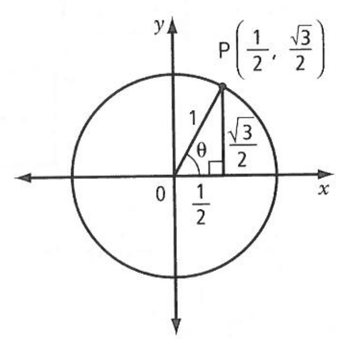


These special triangles can be scaled to fit within the unit circle ($r = 1$).

$\theta = \frac{\pi}{4}$



$\theta = \frac{\pi}{3}$



Working Example 1: Determine Coordinates for a Point on the Unit Circle

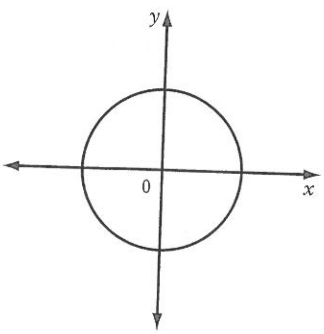
Determine the missing y-coordinate for each point on the unit circle.

a) point A $\left(-\frac{4}{5}, y\right)$ in quadrant II

b) point B $\left(\frac{3}{10}, y\right)$ in quadrant IV

Solution

a) Start by sketching the point on the unit circle. Then, solve for y in the equation of the unit circle.



The equation of the unit circle is $x^2 + y^2 = 1$.
Substitute in the known value of x.

$\square^2 + y^2 = 1$

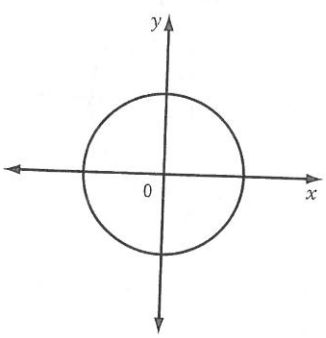
Solve for y.

Isolate y^2. Then, take the square root of both sides.

Since quadrant ________ is specified, take only the ______________ root.
(*positive* or *negative*)

Therefore, $y =$ ________.

b) Start by sketching the point on the unit circle. Then, solve for y in the equation of the unit circle.



$x^2 + y^2 = 1$

Check that your answer is reasonable using the fact that $|y| \leq 1$ on the unit circle.

Since quadrant ________ is specified, take only the ______________ root.
(*positive* or *negative*)

Therefore, $y =$ ________.

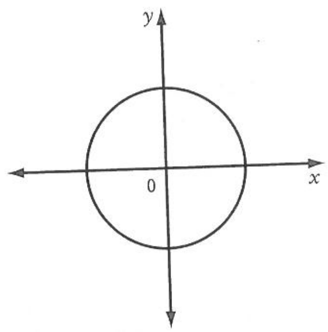
Working Example 2: Reflections of $\frac{\pi}{6}$ on the Unit Circle

Determine the coordinates of all points on the unit circle for which the reference angle is $\frac{\pi}{6}$.

Solution

Start by drawing a diagram showing $\frac{\pi}{6}$ reflected in all four quadrants.

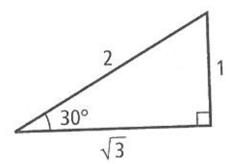
What is $\frac{\pi}{6}$ in degrees?



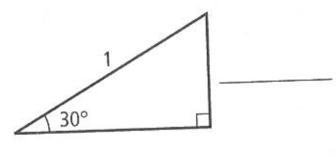
The angles (in radians) are $\frac{\pi}{6}$, $\frac{\square}{\square}$, $\frac{\square}{\square}$, $\frac{\square}{\square}$.

The points are A$\left(\frac{\pi}{6}\right)$, B$\left(\frac{\square}{\square}\right)$, C$\left(\frac{\square}{\square}\right)$, and D$\left(\frac{\square}{\square}\right)$.

The appropriate special triangle is:



Scaled down to fit within a unit circle ($r = 1$):



Therefore, point A$\left(\frac{\pi}{6}\right)$ has coordinates ______________.

- Point B$\left(\frac{\square}{\square}\right)$ is a reflection in the y-axis of point A.

  The x-coordinate changes sign and the y-coordinate stays the same.

  Therefore, point B has coordinates ______________.

- Point C$\left(\frac{\square}{\square}\right)$ is a reflection in the x-axis and y-axis of point A.

  The x-coordinate is ______________ and the y-coordinate is ______________.

  Therefore, point C has coordinates ______________.

- D$\left(\frac{\square}{\square}\right)$ is a reflection in the x-axis of point A.

  The x-coordinate is ______________ and the y-coordinate is ______________.

  Therefore, point D has coordinates ______________.

Check Your Understanding

Practise

1. Determine the equation of a circle centred at (0, 0) with each radius.
   a) 25 units
   b) 1.1 units

2. Is each point on the unit circle? Give evidence to support your answer.
   a) (0.65, −0.76)

   The equation of the unit circle is ______________.

Left Side	Right Side

   Conclusion: ______________________

   b) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

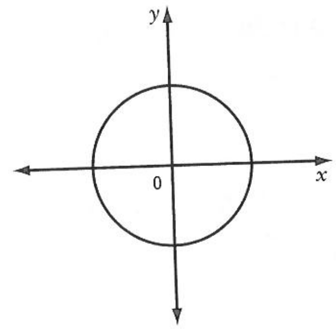
   c) $\left(\frac{\sqrt{7}}{2}, -\frac{1}{7}\right)$

> What information could you use to answer part c) other than a left side/right side proof?

3. Determine the missing coordinate for each point on the unit circle. Draw a diagram to support your answer.

a) point $A\left(x, \frac{5}{13}\right)$ in quadrant II

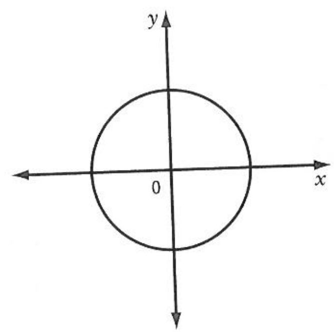
$x^2 + y^2 = 1$



Since quadrant ________ is specified, take only the ________________ root.
(*positive* or *negative*)

Therefore, $x =$ ________.

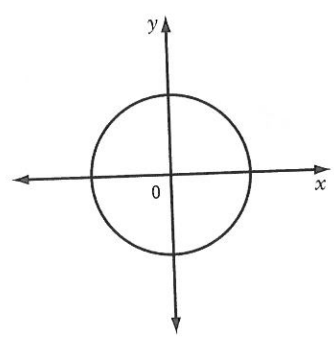
b) point $B\left(\frac{1}{6}, y\right)$ in quadrant IV



Since quadrant ________ is specified, take only the ________________ root.
(*positive* or *negative*)

Therefore, $y =$ ________.

c) point $C\left(x, -\frac{1}{2}\right)$ in quadrant III

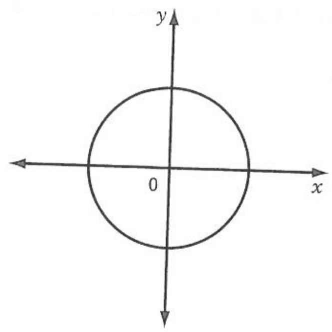


Do you recognize this pair of values? What angle is associated with it?

4. If P(θ) is the point at which the terminal arm of angle θ in standard position intersects the unit circle, determine the exact coordinates of each of the following.

The word *exact* in the question is a clue to use special triangles.

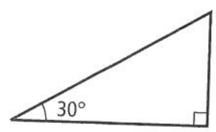
a) $P\left(\frac{\pi}{2}\right)$



b) $P(2\pi)$

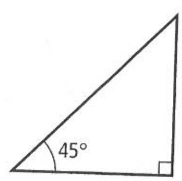
c) $P\left(\frac{2\pi}{3}\right)$

d) $P\left(\frac{5\pi}{4}\right)$



e) $P\left(-\frac{\pi}{4}\right)$

f) $P\left(\frac{23\pi}{6}\right)$



5. Determine the value of angle θ in standard position, $0 \le \theta < 2\pi$, given the coordinates of $P(\theta)$, the point at which the terminal arm intersects the unit circle.

The domain is given in radians, so your answers should also be in radians.

a) $P(\theta) = (-1, 0)$ $\theta =$ ________

b) $P(\theta) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ $\theta =$ ________

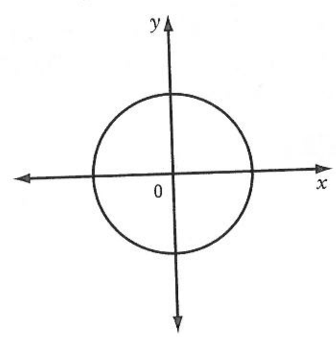
c) $P(\theta) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\theta =$ ________

d) $P(\theta) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $\theta =$ ________

This question should help you complete #5 and #6 on page 187 of *Pre-Calculus 12*.

6. Determine the arc length on the unit circle from (1, 0) to each point.

a) $P\left(\frac{\pi}{2}\right)$

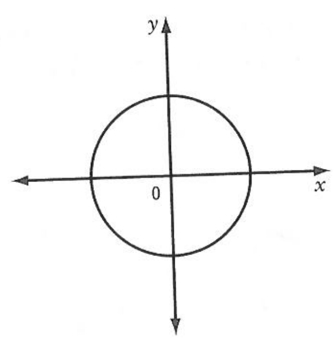


$\theta =$ ________

On the unit circle, $r = 1$.

So, $a =$ ________.

b) $P(\theta)\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



$\theta =$ ________

On the unit circle, $r = 1$.

So, $a =$ ________.

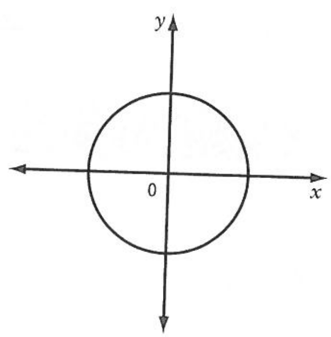
Apply

7. Earth moves in a roughly circular orbit around the sun at a distance of approximately 150 000 000 km. To estimate longer distances in space, scientists measure in multiples of the Earth–sun radius, or astronomical units (1 AU = 149 597 870.691 km). NASA defines 1 AU as the radius of an unperturbed circular orbit of a massless theoretical body revolving about the sun in $\frac{2\pi}{k}$ days, where k is a constant exactly equal to 0.017 202 098 95.

   a) Placing the sun at the origin, write an equation representing Earth's orbit, in kilometres, assuming a circular orbit.

   b) Placing the sun at the origin, write an equation representing Earth's orbit (in AU), assuming a circular orbit.

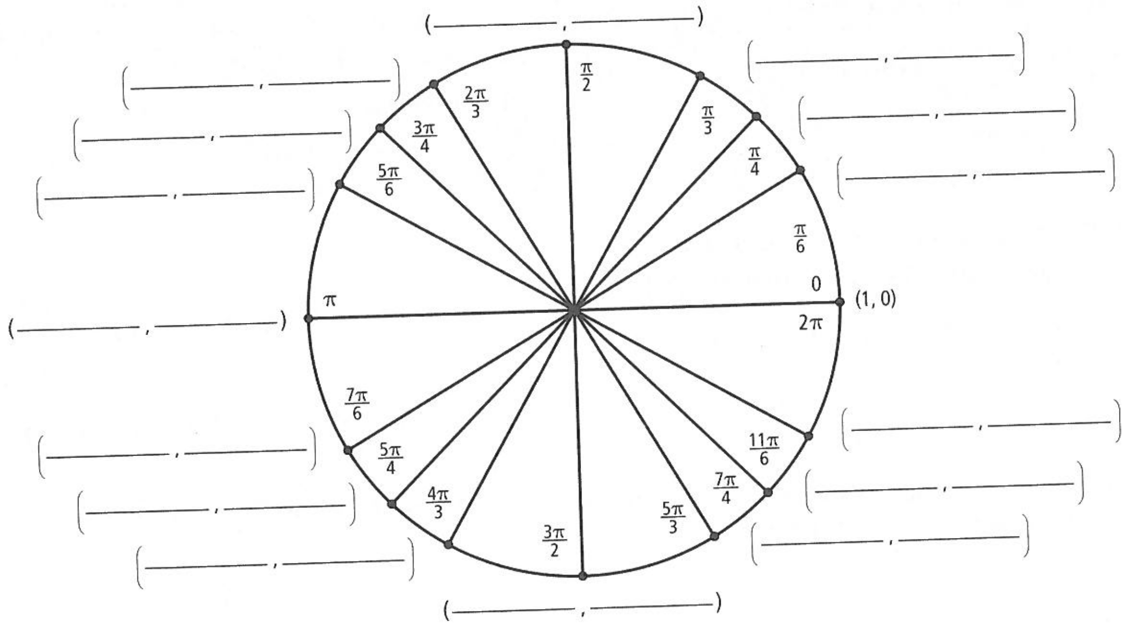
   c) Mars is 1.38 AU from the sun. If the unit circle shown at right describes Earth's orbit, sketch the orbit of Mars.

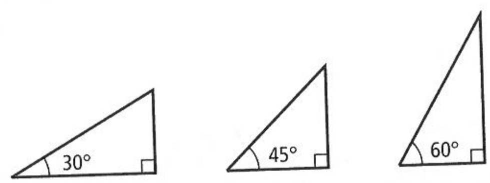


   d) One year on Mars represents one complete rotation. How long (in radians) is 1 Mars-year on Earth? How long is this in Earth-days?

Connect

8. Give the exact coordinates of each P(θ) listed on the diagram below.





Check your work by referring to the Key Ideas on page 186 of *Pre-Calculus 12*.