4.3 Trigonometric Ratios

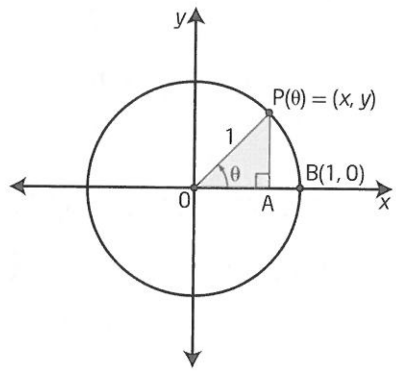
KEY IDEAS

- These are the primary trigonometric ratios:

sine	cosine	tangent
$\sin \theta = \frac{y}{r}$	$\cos \theta = \frac{x}{r}$	$\tan \theta = \frac{y}{x}$

- For points on the unit circle, $r = 1$. Therefore, the primary trigonometric ratios can be expressed as:

$\sin \theta = \frac{y}{1} = y$ $\quad \cos \theta = \frac{x}{1} = x$ $\quad \tan \theta = \frac{y}{x}$

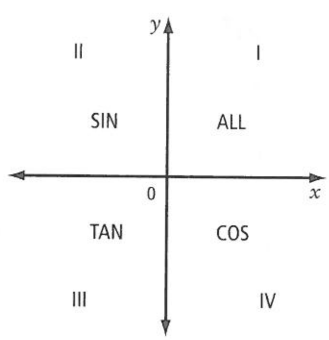


- Since $\cos \theta$ simplifies to x and $\sin \theta$ simplifies to y, you can write the coordinates of $P(\theta)$ as $P(\theta) = (\cos \theta, \sin \theta)$ for any point $P(\theta)$ at the intersection of the terminal arm of θ and the unit circle.

- These are the reciprocal trigonometric ratios:

cosecant	secant	cotangent
$\csc \theta = \frac{1}{\sin \theta}$	$\sec \theta = \frac{1}{\cos \theta}$	$\cot \theta = \frac{1}{\tan \theta}$
$\csc \theta = \frac{r}{y}$	$\sec \theta = \frac{r}{x}$	$\cot \theta = \frac{x}{y}$

- Recall from the CAST rule that
  - $\sin \theta$ and $\csc \theta$ are positive in quadrants I and II
  - $\cos \theta$ and $\sec \theta$ are positive in quadrants I and IV
  - $\tan \theta$ and $\cot \theta$ are positive in quadrants I and III



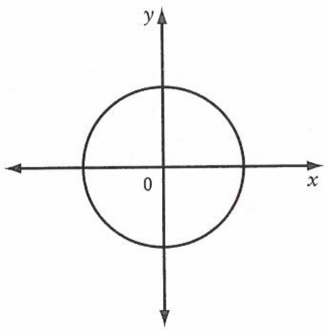
Working Example 1: Determine the Trigonometric Ratios for Angles in the Unit Circle

Point A $\left(-\frac{12}{13}, \frac{5}{13}\right)$ is on the unit circle and on the terminal arm of an angle θ in standard position. Determine the values of the six trigonometric ratios for angle θ.

Solution

Method 1: Use the Unit Circle

Start by sketching point A on the unit circle.



Since point A is on the unit circle,

- the ________-coordinate is defined as $\cos\theta$
- the y-coordinate is defined as ________

The primary trigonometric ratios are

$\sin\theta =$ ________ $\cos\theta =$ ________ $\tan\theta = \frac{y}{x}$

$=$ ________

$=$ ________

Now, take the reciprocal of each ratio to determine the reciprocal trigonometric ratios.

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\square}$ ________ $\theta = \frac{1}{\tan\theta}$

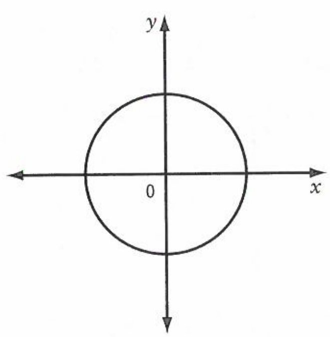
$=$ ________ $=$ ________ $=$ ________

$=$ ________ $=$ ________ $=$ ________

> Leave your answer as a fraction in lowest terms, unless "approximate" (meaning decimal) value is specified.

Method 2: Use a Right Triangle

Start by sketching point A on the unit circle. Draw a vertical line from point A to the x-axis to form right $\triangle ABO$.



The angle is in quadrant ________.

$x =$ ________ $y =$ ________ $r = 1$

> How could you determine r for a point not on the unit circle?

Substitute these values into the definitions of the primary trigonometric ratios.

$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

$=$ ________ $=$ ________ $=$ ________

Take the reciprocal of each ratio to determine the reciprocal trigonometric ratios.

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

$=$ ________ $=$ ________ $=$ ________

Note that this process can be followed for points that are not on the unit circle.

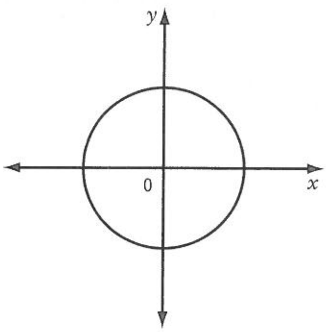
Working Example 2: Exact and Approximate Values for Trigonometric Ratios

a) Determine the exact value of csc $\frac{5\pi}{4}$.

b) Determine the value of cot (-0.5) to four decimal places.

Solution

a) Sketch the angle $\frac{5\pi}{4}$ in standard position on the unit circle.



Point $P\left(\frac{5\pi}{4}\right)$ is in quadrant ______. The reference angle for $\frac{5\pi}{4}$ is $\theta_R =$ ______.

Sketch the special triangle for the reference angle. Determine the values of x ($\cos \theta_R$) and y ($\sin \theta_R$).

For the reference angle θ_R (in quadrant I), $P\left(\frac{\pi}{4}\right) =$ (______, ______).

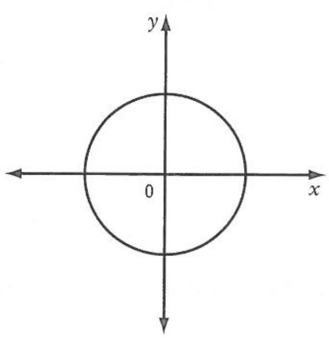
Therefore, for the angle $\frac{5\pi}{4}$, $P\left(\frac{5\pi}{4}\right) =$ (______, ______).

csc θ is the reciprocal of ______.

csc $\theta =$ ______.

> The expression *exact value* is a clue that you should be thinking in terms of special triangles and fractions of π.

b) Draw the angle -0.5 in standard position on the unit circle.



-0.5 is in quadrant ______. The reference angle

θ_R is ______.

cot θ is the reciprocal of ______.

> Is the angle in radians or degrees? Make sure your calculator is on the correct setting.

This is not one of the special angles for which you know exact values of x and y using special triangles. Therefore, calculate the primary trigonometric ratio of the reference angle using your calculator.

Then, take the reciprocal.

Determine the sign (+ or −).

Final answer: cot $(-0.5) \approx$ ______

> Your scientific or graphing calculator has a reciprocal button, usually labelled $\frac{1}{x}$ or x^{-1}. Do not round until the final answer.

Working Example 3: Determine Angles Given Their Trigonometric Ratios

Determine the measure of all angles that satisfy the following conditions. Give exact answers where possible. Otherwise, round to two decimal places.

a) $\sec \theta = \sqrt{2}$ in the domain $0 \leq \theta < 4\pi$

b) $\csc \theta = -1.5557$ in the domain $-360° \leq \theta < 360°$

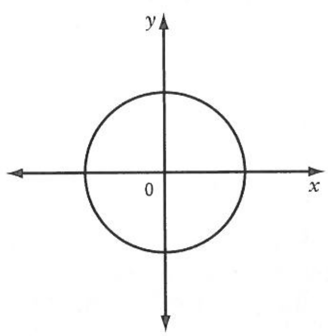
Solution

Degrees or radians?

a) $\sec \theta = \sqrt{2}$. Therefore, $\cos \theta =$ ________.

Draw the special triangle.

Then, draw the angle on the unit circle.



The reference angle θ_R is ________.

Cosine and secant are positive in quadrants ________ and ________.

Add the associated angles to your diagram of the unit circle.

quadrant I angle: ________

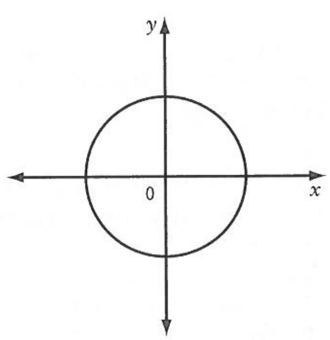
quadrant IV angle: ________

Now, check the domain and determine all additional relevant coterminal angles.

Therefore, $\sec \theta = \sqrt{2}$ when $\theta =$ ________, ________, ________, and ________, $0 \leq \theta < 4\pi$.

b) $\csc \theta = -1.5557$. Therefore, $\sin \theta =$ ________.

Draw the angle on the unit circle and calculate θ_R.



The reference angle θ_R is ________.

Sine and cosecant are negative in quadrants ________ and ________.

Add the associated angles to your diagram of the unit circle.

quadrant III angle: ________

quadrant IV angle: ________

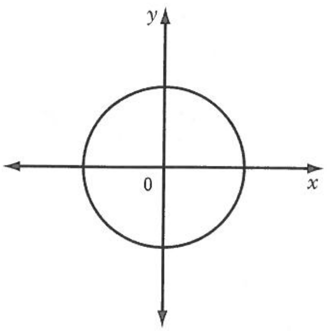
Now, check the domain and determine all additional relevant coterminal angles.

Therefore, $\csc \theta = -1.5557$ when $\theta =$ ________, ________, ________, and ________, $-360° \leq \theta < 360°$.

Check Your Understanding

Practise

1. Point P$\left(\frac{7}{25}, -\frac{24}{25}\right)$ is on the unit circle and on the terminal arm of an angle θ in standard position. Determine the values of the six trigonometric ratios for angle θ.

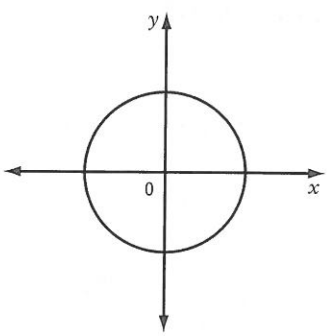


sin θ = ________ cos θ = ________ tan θ = ________

csc θ = ________ sec θ = ________ cot θ = ________

2. Without using a calculator, determine the sign (+ or −) of each of the following.

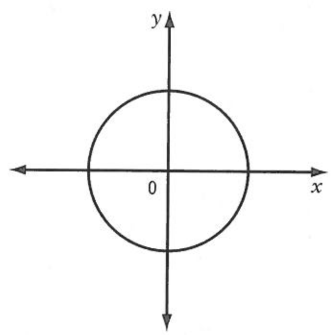
a) sin 580°



quadrant ________

sign is ________

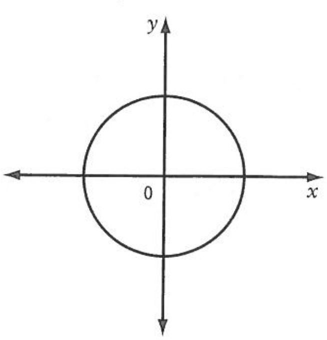
b) tan 1



quadrant ________

sign is ________

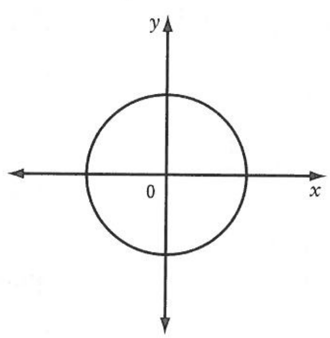
c) $\csc \theta = \frac{2\pi}{3}$



quadrant ________

sign is ________

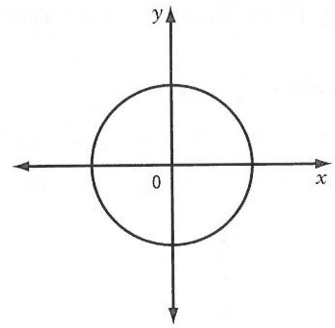
d) $\sec \theta = \frac{5\pi}{4}$



quadrant ________

sign is ________

3. In which quadrant(s) is (are) the terminal arm(s) of angle θ given the following conditions?



a) $\cot \theta$ is positive ________

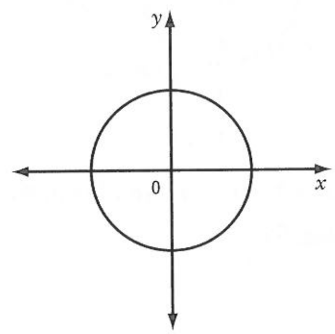
b) $\cot \theta$ is positive and $\sin \theta$ is negative ________

c) $\csc \theta = 1.2$ ________

d) $\csc \theta = 1.2$ and $\cos \theta = -0.574$ ________

4. What is the exact value for each trigonometric ratio?

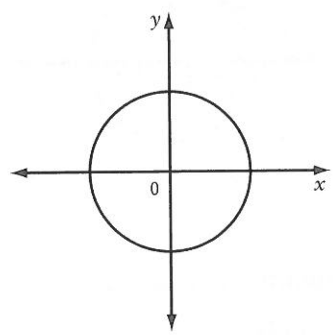
a) $\cos \frac{\pi}{3}$



$P\left(\frac{\pi}{3}\right)$ is in quadrant ________.

$\theta_R =$ ________

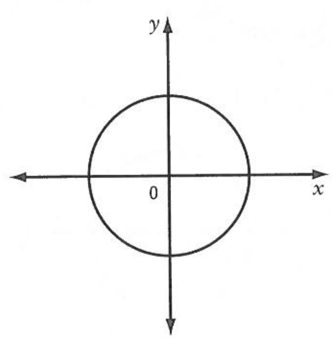
b) $\sin \frac{\pi}{4}$



$P\left(\frac{\pi}{4}\right)$ is in quadrant ________.

$\theta_R =$ ________

c) $\tan 3\pi$

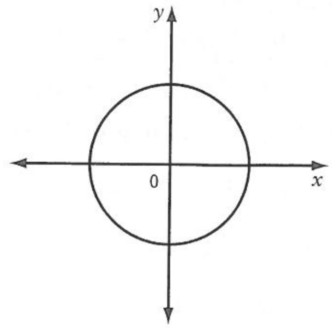


$P(3\pi)$ is a quadrantal angle.

d) $\cot\left(-\frac{2\pi}{3}\right)$

$P\left(-\frac{2\pi}{3}\right)$ is in quadrant ________.

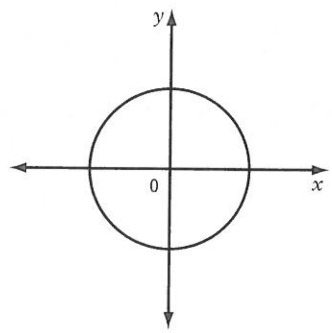
θ_R = ________



e) $\sec\frac{5\pi}{6}$

$P\left(\frac{5\pi}{6}\right)$ is in quadrant ________.

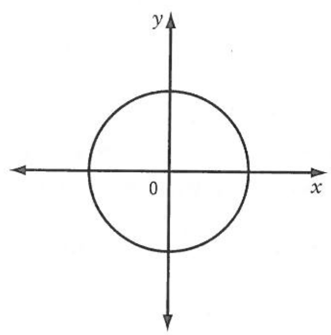
θ_R = ________



f) $\csc\left(-\frac{9\pi}{4}\right)$

$P\left(-\frac{9\pi}{4}\right)$ is in quadrant ________.

θ_R = ________



5. Determine the approximate value for each trigonometric ratio, to three decimal places.

a) sec 74°

sec θ is the reciprocal of ________

quadrant: ________

sign (+ or −): ________

b) cot 104°

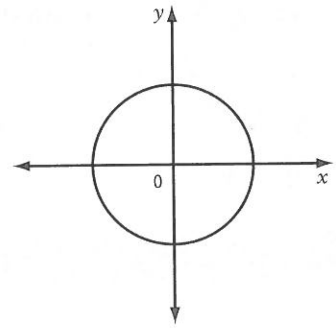
c) csc 2.8

d) $\sec\left(-\frac{7\pi}{10}\right)$

These questions are similar to #1 and #2 on page 201 of *Pre-Calculus 12*.

Apply

6. Determine the measure of all angles that satisfy the following conditions. Round your answers to the nearest degree.



a) $\tan\theta = -3.078$ in the domain $0° \le \theta < 720°$

$\theta_R = \tan^{-1}(+3.078)$

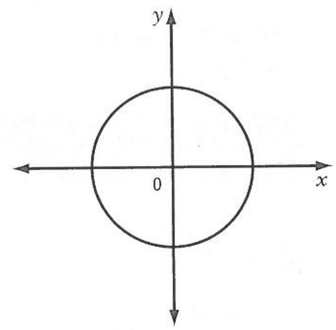
$\approx$ ________

Tangent is negative in quadrants ________ and ________.

Therefore, $\tan\theta = -3.078$ when $\theta \approx$ ________,

________, ________, and ________, $0° \le \theta < 720°$.

Which other coterminal angles fall within the domain?



b) $\sec\theta = -1.046$ in the domain $-360° \le \theta < 360°$

$\sec\theta$ is the reciprocal of ________.

$\theta_R \approx$ ________

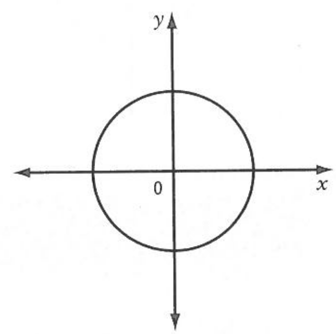
Secant is negative in quadrants ________ and ________.

Therefore, $\sec\theta = -1.046$ when $\theta \approx$ ________,

________, ________, and ________, $-360° \le \theta < 360°$.

7. Determine the measure of all angles that satisfy the following conditions. Give exact answers.

Drawing the special triangle may help.



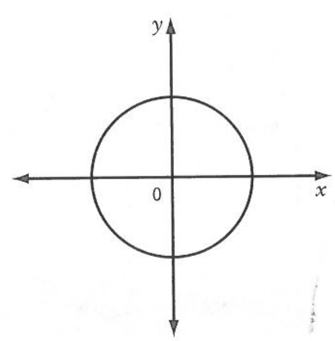
a) $\sin\theta = -\frac{\sqrt{3}}{2}$ in the domain $0 \le \theta < 4\pi$

$\theta_R =$ ________

Sine is negative in quadrants ________ and ________.

Therefore, $\sin\theta = -\frac{\sqrt{3}}{2}$ when $\theta =$ ________, ________,

________, and ________, $0 \le \theta < 4\pi$.



b) $\csc\theta = 2$ in the domain $-2\pi \le \theta < 2\pi$

What is the reciprocal of 2?

$\csc\theta$ is the reciprocal of ________.

$\theta_R =$ ________

Cosecant is positive in quadrants ________ and ________.

Therefore, $\csc\theta = 2$ when $\theta =$ ________, ________,

________, and ________, $-2\pi \le \theta < 2\pi$.

8. Determine the value of the five other trigonometric ratios if $\csc \theta = \frac{5}{3}$, $90° \leq \theta < 180°$.

The angle is in quadrant ________.

$x =$ ________ $y =$ ________ $r =$ ________

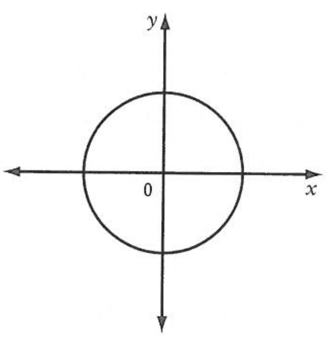
$\sin \theta =$ ________ $\cos \theta =$ ________ $\tan \theta =$ ________

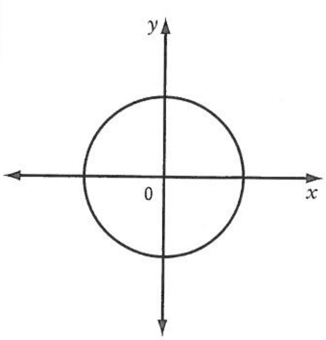
$\sec \theta =$ ________ $\cot \theta =$ ________

This question will help you with #12 on page 202 of *Pre-Calculus 12*.

Connect

9. Choose any two of the special angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$. Complete the table below. You may also choose the quadrantal angles (on the axes), but then you will have to change the headings on the table.

$\theta_R =$	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
	sin csc			
	cos sec			
	tan cot			

$\theta_R =$	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
	sin csc			
	cos sec			
	tan cot			